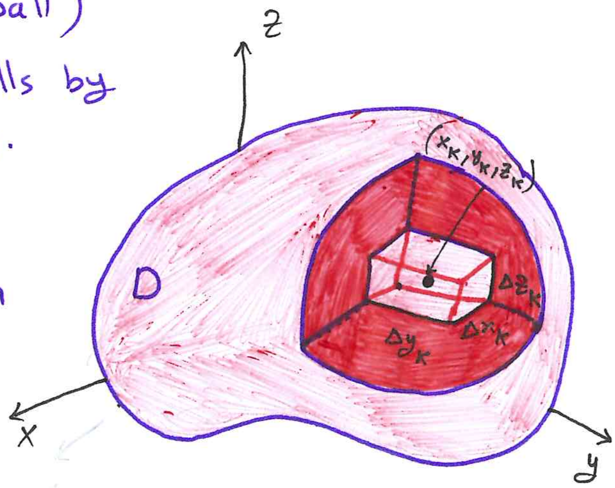
15.5 Triple Integrals in Rectangular Coordinates

How to construct triple integral?

- Let $F(x,y,z)$ be a function defined on closed bounded region D in space. (D can be a solid ball)
- we partion D into rectangular cells by planes // to the coordinate axes.
- we number the cells that lie completely inside D from $1, 2, \cdots, n$
- Choose any point (x_K, y_K, z_K) in the K^{th} cell whose dimensions are Δx_K by Δy_K by Δz_K.



2] • The volume of the K^{th} cell is $\Delta V_K = \Delta x_K \, \Delta y_K \, \Delta z_K$

- The volume of D is approximated by $S_n = \sum_{K=1}^{n} \Delta V_K$ when $F=1$
- As $\Delta x_K, \Delta y_K, \Delta z_K \longrightarrow 0$, the volume of a closed bounded region D in space is $V = \iiint_D dV$

I] • The sum is $S_n = \sum_{K=1}^{n} F(x_K, y_K, z_K) \, \Delta V_K$

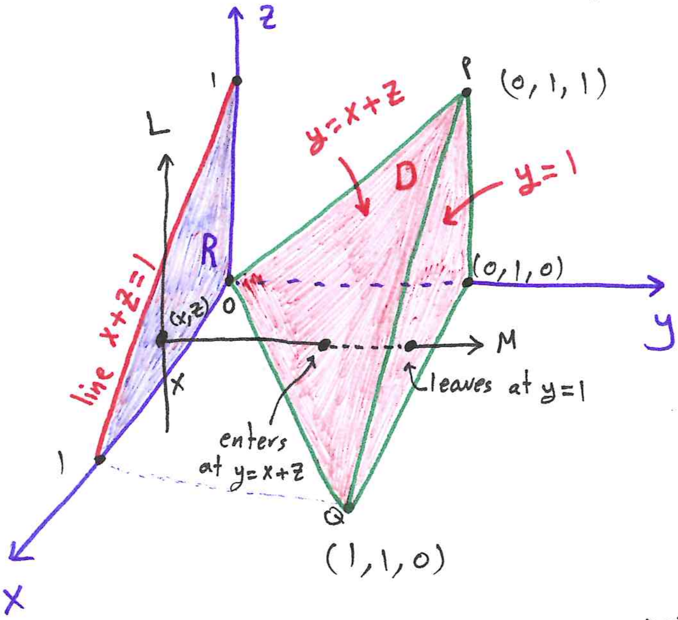
$\Rightarrow$ As $\|P\| = \{\max \Delta x_K, \Delta y_K, \Delta z_K\} \longrightarrow 0$ and if the limit is attained (F is continuous on smooth boundary) then we say F is integrable on D. We call this limit triple integral of F on D:

$$\lim_{n \to \infty} S_n = \iiint_D F(x,y,z)\, dV \quad \text{or} \quad \lim_{\|P\| \to 0} S_n = \iiint_D F(x,y,z)\, dx\, dy\, dz$$

Now we can calculate volumes of solids enclosed by curved surfaces.

Exp [A] Use the order $dy\,dz\,dx$ to find the volume of the tetrahedron D with vertices $(0,0,0), (1,1,0), (0,1,0), (0,1,1)$. (120)

$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^{1} dy\,dz\,dx$$



[1] * We sketch D along with its shadow R in the xz-plane.

[2] * The right-hand bounding surface of D lies in the plane $y=1$

The left-hand bounding surface of D lies in the plane $y=x+z$

since $\vec{OP} = \vec{j} + \vec{k}$ and $\vec{OQ} = \vec{i} + \vec{j}$ and $\vec{OP} \times \vec{OQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\vec{i} + \vec{j} - \vec{k}$

$\Rightarrow$ The plane is $-x+y-z=0 \Leftrightarrow y=x+z$

[3] * The upper boundary of R is the line $z=1-x$.
The lower boundary of R is the line $z=0$.

[4] * First we find y-limits of integration: we draw a line M through a point (x,z) in R // to y-axis. The line enters D at $y=x+z$ and leaves at $y=1$.

2^{nd}: we find z-limits: we draw a line L through (x,z) // to z-axis enters R at $z=0$ and leaves at $z=1-x$

3^{rd}: we find x-limits: As L sweeps across R, the value of x varies from $x=0$ to $x=1$.

$$= \int_0^1 \int_0^{1-x} \int_{x+z}^{1} dy\,dz\,dx = \int_0^1 \int_0^{1-x} (1-x-z)\,dz\,dx = \int_0^1 \left(\frac{x^2}{2} - x + \frac{1}{2}\right) dx = \frac{1}{6}$$

[B] Find this volume using the order $dz\,dy\,dx$.

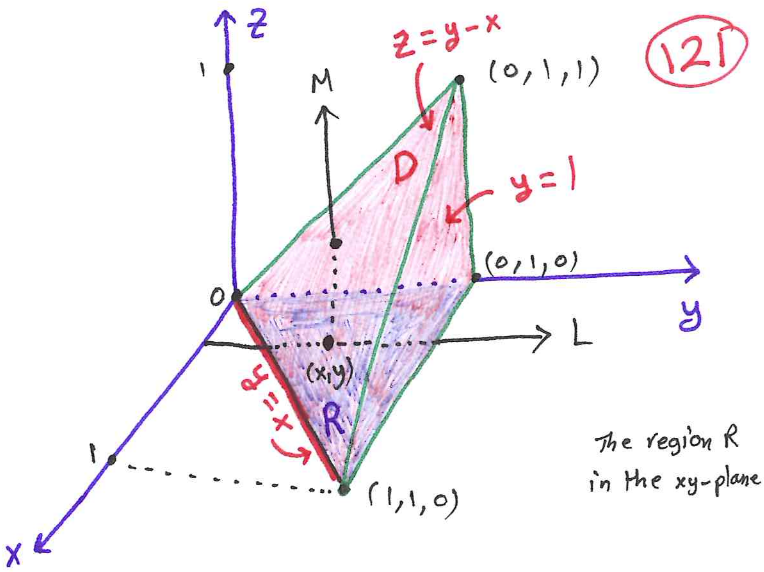
In this case, shadow region R is in the xy-plane

- The line M // z-axis and
- The line L // y-axis

$$V=\int_0^1\int_x^1\int_0^{y-x} dz\,dy\,dx$$

$$=\int_0^1\int_x^1 (y-x)\,dy\,dx$$

$$=\int_0^1\left(\frac{x^2}{2}-x+\frac{1}{2}\right)=\frac{1}{6}$$

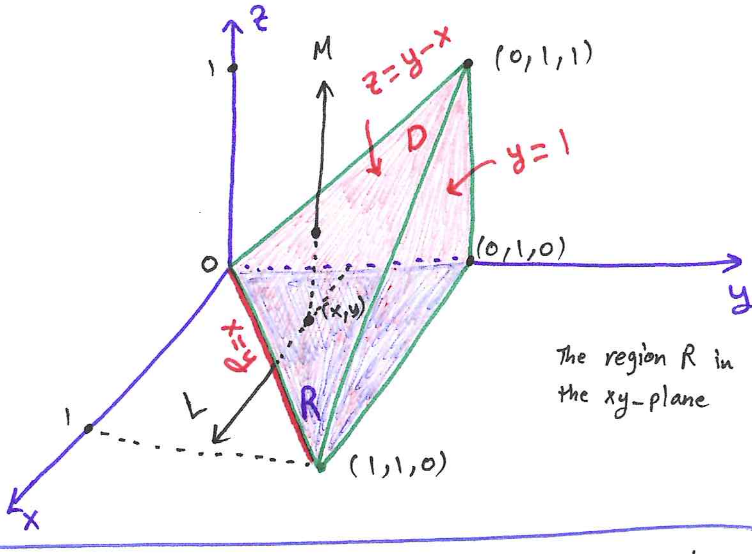


$\boxed{c}$ Find the same volume in the order $dz\,dx\,dy$

$$V=\int_0^1\int_0^y\int_0^{y-x} dz\,dx\,dy$$

$$=\int_0^1\int_0^y (y-x)\,dx\,dy$$

$$=\int_0^1 \frac{y^2}{2}\,dy = \frac{1}{6}$$

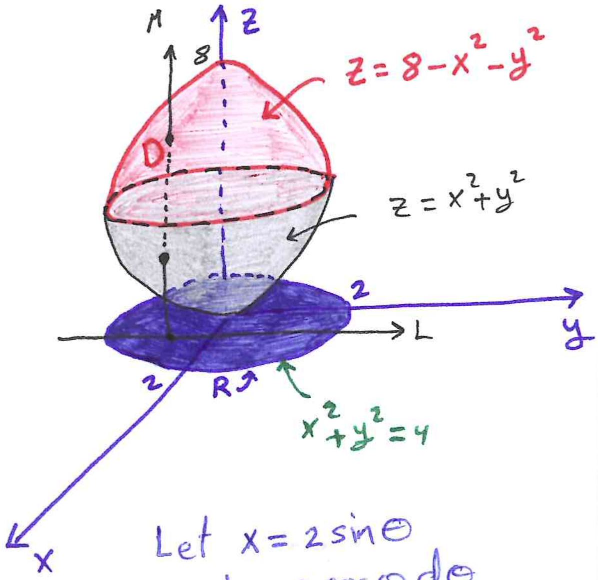


<u>Exp</u> Let D be the region bounded by the paraboloids $z=8-x^2-y^2$ and $z=x^2+y^2$. Write six different triple integrals for the volume of D. Evaluate one of the integrals.

- $8-x^2-y^2=x^2+y^2 \Leftrightarrow x^2+y^2=4$
- $$V=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2(4-x^2-y^2)\,dy\,dx$$

$$=4\int_{-2}^{2}\left(4\sqrt{4-x^2}-x^2\sqrt{4-x^2}-\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right)dx$$



Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2}=2\cos\theta$

$\Rightarrow$

$$\Rightarrow V = \frac{128}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta \, d\theta = \frac{32}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta = 16\pi$$ 122

or $$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= 8\int_0^2\int_0^{\sqrt{4-x^2}} (4-x^2-y^2)\,dy\,dx = 8\int_0^{\frac{\pi}{2}}\int_0^2 (4-r^2)\,r\,dr\,d\theta$$

$$= 32\int_0^{\frac{\pi}{2}} d\theta = 16\pi$$

- $$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dx\,dy$$

- $$V = \int_{-2}^{2}\int_{y^2}^{4}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dz\,dy + \int_{-2}^{2}\int_{4}^{8-y^2}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dz\,dy$$ M // x-axis, L // z-axis

because the region R now is in yz-plane "x=0" $\Rightarrow z_1 = y^2$, $\Rightarrow z_2 = 8-y^2$

They intersect when $z = 8-x^2-y^2 = 8-z$ $\Rightarrow z = 4$

- $$V = \int_0^4\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dy\,dz + \int_4^8\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dy\,dz$$ M // x-axis, L // y-axis

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- $$V = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dz\,dx + \int_{-2}^{2}\int_{4}^{8-x^2}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dz\,dx$$ M // y-axis, L // z-axis

because the region R now is in xz-plane "y=0" $\Rightarrow z_1 = x^2$, $\Rightarrow z_2 = 8-x^2$, $\Rightarrow z = 4$

- $$V = \int_0^4\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dx\,dz + \int_4^8\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dx\,dz$$ M // y-axis, L // x-axis

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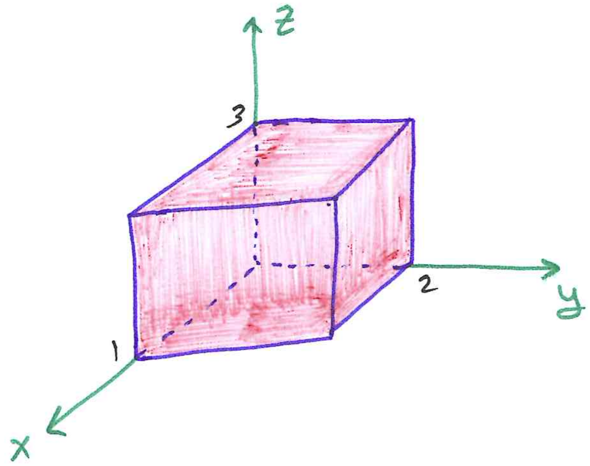
* The average value of a function F over a region D in space is defined by:

$$av(F) = \frac{1}{\text{volume of } D} \iiint_D F \, dv$$

Exp 1. Find the volume of rectangular solid in the first octant bounded by the coordinate planes and the planes $x=1$, $y=2$, $z=3$.

$$V = \int_0^3 \int_0^1 \int_0^2 dy\, dx\, dz = 6$$



2. Write six different iterated triple integrals for the volume.

$$V = \int_0^1 \int_0^3 \int_0^2 dy\, dz\, dx$$

$$= \int_0^2 \int_0^3 \int_0^1 dx\, dz\, dy$$

$$= \int_0^3 \int_0^2 \int_0^1 dx\, dy\, dz$$

$$= \int_0^2 \int_0^1 \int_0^3 dz\, dx\, dy$$

$$= \int_0^1 \int_0^2 \int_0^3 dz\, dy\, dx$$

3. Find the average value of $F(x,y,z) = xyz$ throughout this rectangular region.

$$av(F) = \frac{1}{6} \int_0^3 \int_0^2 \int_0^1 xyz \, dx\, dy\, dz = \frac{1}{6} \int_0^3 \int_0^2 \frac{yz}{2}\, dy\, dz$$

$$= \frac{1}{12} \int_0^3 2z\, dz = \frac{9}{12} = \frac{3}{4}.$$