

# 11.2 Calculus with Parametric Curves

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\* Parametric formulars for  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  : • Given the parametric equations:

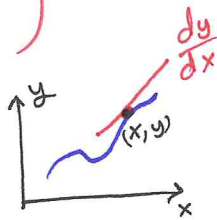
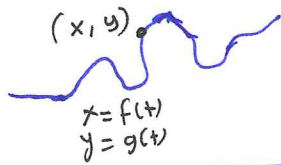
$$\dot{y} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\ddot{y} = \frac{d^2y}{dx^2} = \frac{d\dot{y}/dt}{dx/dt}$$

equations:

$$x = f(t), y = g(t).$$

• If  $x, y, \dot{y}$  are differentiable at  $t$  with  $\frac{dx}{dt} \neq 0$ , then \* holds at any point  $t$ .



Exp • Find the tangent to the curve  $x=t, y=\sqrt{t}, t=\frac{1}{4}$

• The point is  $(f(\frac{1}{4}), g(\frac{1}{4})) = (\frac{1}{4}, \frac{1}{2})$

• The tangent line is  $y - \frac{1}{2} = m(x - \frac{1}{4})$ , where

$$\text{the slope } m = \left. \frac{dy}{dx} \right|_{t=\frac{1}{4}} = \left. \frac{dy/dt}{dx/dt} \right|_{t=\frac{1}{4}} = \left. \frac{\frac{1}{2\sqrt{t}}}{1} \right|_{t=\frac{1}{4}} = 1$$

$\Rightarrow$  The tangent line becomes  $y - \frac{1}{2} = x - \frac{1}{4}$

$$y = x + \frac{1}{4}$$

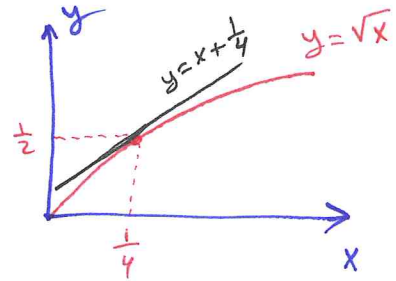
• Find  $\frac{d^2y}{dx^2}$  as a function of  $t$

$$\frac{dy}{dx} = \dot{y} = \frac{dy/dt}{dx/dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{d^2y}{dx^2} = \ddot{y} = \frac{d\dot{y}/dt}{dx/dt} = \frac{\frac{1}{2}(-\frac{1}{2})t^{-\frac{3}{2}}}{1}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{4\sqrt{t^3}}$$

$$\text{• Find } \left. \frac{d^2y}{dx^2} \right|_{t=\frac{1}{4}} = \frac{-1}{4\sqrt{\frac{1}{64}}} = \frac{-1}{4 \cdot \frac{1}{8}} = \frac{-1}{\frac{1}{2}} = -2$$



Exp Find the slope of the curve

$$x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4 \quad \text{at } t=2$$

"x and y are implicitly differentiable"

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• Note that when  $t=2 \Rightarrow x^3 + 2(2)^2 = 9 \Rightarrow x^3 + 8 = 9$

$$\Rightarrow x^3 = 1$$

$$\Rightarrow x = 1$$

$$\Rightarrow 2y^3 - 3(2)^2 = 4 \Rightarrow 2y^3 - 12 = 4$$

$$\Rightarrow 2y^3 = 16$$

$$\Rightarrow y = 2$$

• The slope is:

$$m = \frac{dy/dt}{dx/dt} \Big|_{t=2}$$

$$= \frac{\frac{t}{y^2}}{\frac{-4t}{3x^2}} \Big|_{t=2}$$

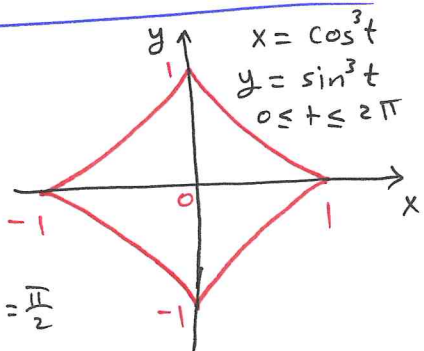
$$= \frac{\frac{2}{4}}{\frac{-8}{3}} = -\frac{1}{2} \cdot \frac{3}{8} = -\frac{3}{16}$$

$$3x^2 \frac{dx}{dt} + 4t = 0 \Rightarrow \frac{dx}{dt} = \frac{-4t}{3x^2}$$

$$6y^2 \frac{dy}{dt} - 6t = 0 \Rightarrow \frac{dy}{dt} = \frac{t}{y^2}$$

Exp Find the area enclosed by the astroid

$$A = 4 \int_0^1 y \, dx = 4 \int_{\frac{\pi}{2}}^0 \sin^3 t (3 \cos^2 t (-\sin t)) \, dt$$



when  $x=0 \Rightarrow 0 = \cos^3 t \Rightarrow 0 = \cos t \Rightarrow t = \cos^{-1} 0 = \frac{\pi}{2}$

$x=1 \Rightarrow 1 = \cos^3 t \Rightarrow 1 = \cos t \Rightarrow t = \cos^{-1} 1 = 0$

$$A = 12 \int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t \, dt = 12 \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos 2t}{2}\right)^2 \left(\frac{1 + \cos 2t}{2}\right) \, dt$$

$$= \frac{12}{8} \int_0^{\frac{\pi}{2}} (1 - 2\cos 2t + \cos^2 2t)(1 + \cos 2t) \, dt$$

$$= \frac{3}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2t - \cos^2 2t + \cos^3 2t) \, dt = \dots = \frac{3\pi}{8}$$

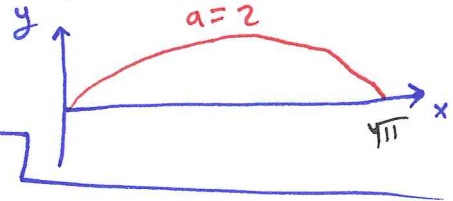
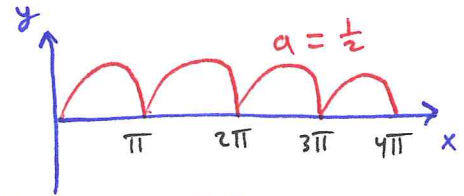
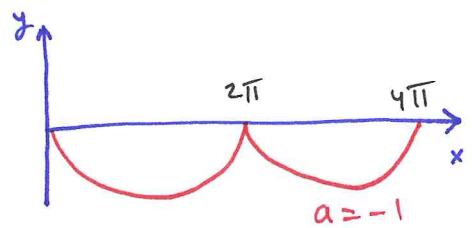
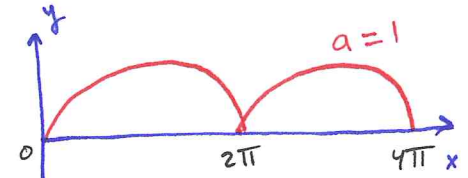
Exp Find the area under one arch of the cycloid:

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$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

when  $a = 1$ .

$$\begin{aligned} A &= \int_0^{2\pi} y \, dx = \int_0^{2\pi} (1 - \cos t)(1 - \cos t) \, dt \\ &= \int_0^{2\pi} (1 - \cos t)^2 \, dt = \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) \, dt \\ &= \int_0^{2\pi} \left( 1 - 2\cos t + \frac{1 + \cos 2t}{2} \right) \, dt \\ &= 3\pi \end{aligned}$$



Exp Find the area enclosed by the y-axis and the curve  $x = t - t^2, y = 1 + e^{-t}, 0 \leq t \leq 1$

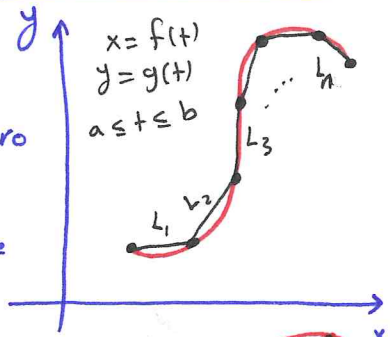
$$\begin{aligned} A &= \int_0^1 x \, dy = \int_0^1 (t - t^2)(-e^{-t}) \, dt = \int_1^0 (t - t^2) e^{-t} \, dt \\ &= \left[ (t^2 - t) e^{-t} - (1 - 2t) e^{-t} + 2 e^{-t} \right]_1^0 = (-1 + 2) - (e^{-1} + 2e^{-1}) \\ &= 1 - \frac{3}{e} \end{aligned}$$

\*Length of a Curve defined parametrically:

- If  $f'$  and  $g'$  are continuous and not simultaneously zero on  $[a, b]$
- the curve traversed exactly once as  $t$  increases on  $[a, b]$

Then the length of the curve is

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \end{aligned}$$



$$\sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$L = \lim_{n \rightarrow \infty} \sum_{k=1}^n L_k$$

Exp Find the length of the curve:

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$$x = \frac{t^2}{2}, \quad y = \frac{(2t+1)^{\frac{3}{2}}}{3}, \quad 0 \leq t \leq 4$$

$$L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^4 \sqrt{t^2 + 2t+1} dt = \int_0^4 \sqrt{(t+1)^2} dt$$

$$= \int_0^4 |t+1| dt = \int_0^4 (t+1) dt = \left(\frac{t^2}{2} + t\right) \Big|_0^4 = 12$$

### Area of Surfaces of Revolution

Given a smooth curve  $x=f(t), y=g(t), a \leq t \leq b$  traversed exactly once as  $t$  increases from  $a$  to  $b$ .

The area of the surfaces generated by revolving the curve about coordinates axes are as follows:

\* Revolution about x-axis ( $y \geq 0$ ):  $S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

\* Revolution about y-axis ( $x \geq 0$ ):  $S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

Exp Find the area of surface generating by revolving the curves

•  $x = \cos t, \quad y = 2 + \sin t, \quad 0 \leq t \leq 2\pi, \quad \underline{x\text{-axis}}$

$$S = \int_0^{2\pi} 2\pi (2 + \sin t) \sqrt{\sin^2 t + \cos^2 t} dt = 2\pi \int_0^{2\pi} (2 + \sin t) dt$$

$$= 2\pi [2t - \cos t] \Big|_0^{2\pi} = 2\pi [4\pi - 1 - (0 - 1)] = 8\pi^2$$