$\boxed{14.7}$ Extreme Values and Saddle Points

* To find local extreme values of $f(x)$, we look for: $\boxed{1}$ local max, $\boxed{2}$ local min $\boxed{3}$ [redacted] critical points where $f'(x)=0$ "the graph has a horizontal tangent line"

* To find local extreme values for $f(x,y)$, we look for $\boxed{1}$ local max, $\boxed{2}$ local min $\boxed{3}$ saddle points where the graph $z=f(x,y)$ has horizontal tangent plane.

Def[1]: Let $f(x,y)$ be defined on region R containing the point (a,b). Then

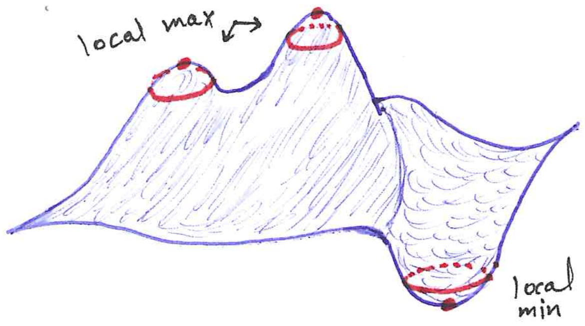
$\boxed{1}$ $f(a,b)$ is **local max** if $f(a,b) \geq f(x,y)$ for all domain points in an open disk centered at (a,b).

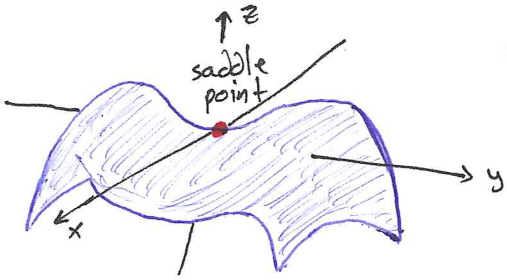
$\boxed{2}$ $f(a,b)$ is **local min** if $f(a,b) \leq f(x,y)$ for all domain points in an open disk centered at (a,b).

Def[3] A diff function $f(x,y)$ has a **saddle point** at a critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and there are domain points (x,y) where $f(x,y) < f(a,b)$.

Def[2] An **interior point** of the domain of a function $f(x,y)$ where $f_x(a,b) = f_y(a,b) = 0$ or one or both of $f_x(a,b)$ and $f_y(a,b)$ DNE is a **critical point** of f.

Th* (First Derivative Test for Local Extreme Values)
If $f(x,y)$ has a local max or local min at an interior point (a,b) of its domain and if the first partial derivative exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.





* Th* says that the only points where $f(x,y)$ can have extreme values are the critical points and boundary points.

Th (Second Derivative Test for local Extreme values)

Suppose that $f, f_x, f_y, f_{xx}, f_{yy}, f_{xy}$ are continuous throughout a disk centered at (a,b) and $f_x(a,b) = f_y(a,b) = 0$. Then,

1. f has local max at (a,b) if $f_{xx}(a,b) < 0$ and $f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b) > 0$.
2. f has local min at (a,b) if $f_{xx}(a,b) > 0$ and $f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b) > 0$.
3. f has a saddle point at (a,b) if $f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b) < 0$.
4. The test is inconclusive at (a,b) if $f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b) = 0$.

---

* The expression $f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ is called the discriminant or Hessian.

Exp Find local max, local min, saddle points of

1. $f(x,y) = x^2 - 4xy + y^2 + 6y + 2$

- $\left.\begin{array}{l} f_x = 2x - 4y = 0 \\ f_y = -4x + 2y + 6 = 0 \end{array}\right\} (2,1)$ is the critical point
- $f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = -4 \quad \Rightarrow f_{xx} f_{yy} - f_{xy}^2 = 4 - 16 = -12 < 0$

  $\Rightarrow (2,1)$ is saddle point

102

$\boxed{2}$ $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

- $f_x = 3x^2 + 6x = 0 \iff 3x(x+2) = 0 \iff x = 0$ or $x = -2$

  $f_y = 3y^2 - 6y = 0 \iff 3y(y-2) = 0 \iff y = 0$ or $y = 2$

- The critical points are $(0,0)$, $(0,2)$, $(-2,0)$, $(-2,2)$
- $f_{xx} = 6x + 6$, $f_{yy} = 6y - 6$, $f_{xy} = 0$
- $\underline{(0,0)}$: $f_{xx}(0,0) = 6$, $f_{yy}(0,0) = -6$, $f_{xy}(0,0) = 0$

  $f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}^2(0,0) = -36 < 0 \Rightarrow (0,0)$ is saddle point
- $\underline{(0,2)}$: $f_{xx}(0,2) = 6$, $f_{yy}(0,2) = 6$, $f_{xy}(0,2) = 0$

  $f_{xx}(0,2)\, f_{yy}(0,2) - f_{xy}^2(0,2) = 36 > 0$ and $f_{xx}(0,2) > 0 \Rightarrow$

  $f(0,2) = -12$ is local min
- $\underline{(-2,0)}$: $f_{xx}(-2,0) = -6$, $f_{yy}(-2,0) = -6$, $f_{xy}(-2,0) = 0$

  $f_{xx}(-2,0)\, f_{yy}(-2,0) - f_{xy}^2(-2,0) = 36 > 0$ and $f_{xx}(-2,0) < 0 \Rightarrow$

  $f(-2,0) = -4$ is local max
- $\underline{(-2,2)}$: $f_{xx}(-2,2) = -6$, $f_{yy}(-2,2) = 6$, $f_{xy}(-2,2) = 0$

  $f_{xx}(-2,2)\, f_{yy}(-2,2) - f_{xy}^2(-2,2) = -36 < 0 \Rightarrow (-2,2)$ is saddle point.

Absolute Max and Absolute Min on Closed Bounded Regions

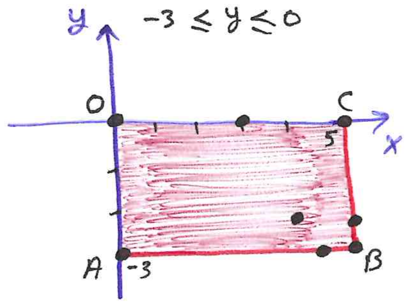
* To find the Absolute Max and Absolute Min of $f(x,y)$ on R:

$\boxed{1}$ Find the interior points of R where f has local max and local min and evaluate f at these points "critical points".

$\boxed{2}$ Find the boundary points of R where f has local max and local min and evaluate f at these points.

$\boxed{3}$ Find Absolute max and Absolute min of $\boxed{1}$ and $\boxed{2}$.

Exp Find the absolute maxima and minima of the function $f(x,y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5$, $-3 \le y \le 0$



- $\underline{OC}$: $f(x,0) = x^2 - 6x + 2$ on $0 \le x \le 5$

  $f' = 2x - 6 = 0 \Rightarrow x = 3$

  $f(3,0) = -7$, $f(0,0) = 2$, $f(5,0) = -3$

- $\underline{CB}$: $f(5,y) = y^2 + 5y - 3$ on $-3 \le y \le 0$

  $f' = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$

  $f(5, -\frac{5}{2}) = -\frac{37}{4}$, $f(5,-3) = -9$, $f(5,0) = -3$

- $\underline{AB}$: $f(x,-3) = x^2 - 9x + 11$ on $0 \le x \le 5$

  $f' = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$

  $f(\frac{9}{2}, -3) = -\frac{37}{4}$, $f(0,-3) = 11$, $f(5,-3) = -9$

- $\underline{AO}$: $f(0,y) = y^2 + 2$ on $-3 \le y \le 0$

  $f' = 2y = 0 \Rightarrow y = 0$

  $f(0,0) = 2$, $f(0,-3) = 11$

- For interior points: $f_x(x,y) = 2x + y - 6 = 0$, $f_y(x,y) = x + 2y = 0$ $\Big\rangle$ $(4,-2)$ is an interior critical point

  $\Rightarrow f(4,-2) = -10$

- The absolute max is 11 at $(0,-3)$

  The absolute min is -10 at $(4,-2)$

Exp Find three numbers whose sum is 9 and whose sum of squares is a minimum.

- $S(x,y,z) = x^2 + y^2 + z^2$ where $x + y + z = 9 \Rightarrow z = 9 - x - y$

  $S(x,y) = x^2 + y^2 + (9-x-y)^2$

- $S_x = 2x - 2(9-x-y) = 0$, $S_y = 2y - 2(9-x-y) = 0$ $\Big\rangle$ $(3,3)$ with $z=3$ is the critical point

- $S_{xx} = 4$, $S_{yy} = 4$, $S_{xy} = 2$ $\Rightarrow S_{xx}(3,3)\,S_{yy}(3,3) - S_{xy}^2(3,3) = 12 > 0$ and $S_{xx} > 0$

  $\Rightarrow$ local min of $S(3,3,3) = 27$