# 10.9 Convergence of Taylor Series:

**Taylor's theorem:** If $f, f', \dots, f'', \dots, f^{(n)}$ are continuous on $[a, b]$ and $f^{(n)}$ is differentiable on $(a, b)$, then there exists $c \in (a, b)$ such that:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(\boxed{c})}{(n+1)!}(b-a)^{n+1}.$$

**Remark:** Notice that the mean value theorem is a special case of Taylor's Theorem, when $n=0$:

$$\Rightarrow f(b) = f(a) + f'(c)(b-a)$$



$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

**Remark:** If we change $b$ by $x$, we get Taylor's formula.

# Taylor's Formula.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)(x-a)^n}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Where $c \in (a, x)$.

$$R_n(x) := \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$ is the Reminder

of order $n$, or the error term resulted from approximating $f$ by $P_n(x)$ on an open Interval $I$ around $a$.

Remark: If $R_n(x) \longrightarrow 0$ as $n \to \infty$, $\forall x \in I$, we say that the taylor series generated by $f$ at $x = a$ converges to $f$ on $I$, and we write:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

**Example:** Show that the Taylor Series generated by $f(x) = e^x$ at $x=0$ converges to $e^x$, $\forall x$.

**sol:** $e^x$ and all derivatives are Cont. on $\mathbb{R}$.

Taylor's formula :

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x).$$

where $R_n(x) = \dfrac{f^{(n+1)}(c)\,(x)^{n+1}}{(n+1)!}$, $c \in (0, x)$.

Thm (10.1)

$$\Rightarrow \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{e^c (x)^{n+1}}{(n+1)!} = e^c \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

$\Rightarrow$ The series converges to $e^x$, $\forall x$

$$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$



**Remark:** Notice that $e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_n(1),\ c \in (0,1)$$

where $R_n(1) = \dfrac{e^c}{(n+1)!} < \dfrac{3}{(n+1)!}$.

$0 < c < 1$
$e^0 < e^c < e^1 < 3$

## Theorem: The Remainder Estimation Theorem:

Given Taylor's Formula: $f(x) = P_n(x) + R_n(x)$.

If there is an $M > 0$ such that $|f^{(n+1)}(t)| \leq M$

for all $t \in (a, x)$, then

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} \right| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \quad \text{......} (*)$$

If $(*)$ holds, for all $n$, then the series Converges

to $f(x)$. (i.e), $f(x) = \sum \frac{f^{(n)}(a)(x-a)^n}{n!}$.

**Example:** Show that the Taylor Series for

$f(x) = \sin x$ at $x = 0$ Converges for all $x$.

**Sol:** $f(x) = \sin x$, $f'(x) = \cos x$

$f''(x) = -\sin x$, $f'''(x) = -\cos x$

$\vdots$ $\qquad\qquad$ $\vdots$

$f^{(2k)}(x) = (-1)^k \sin x$, $f^{(2k+1)}(x) = (-1)^k \cos x$

$\Rightarrow f^{(2k)}(0) = 0$ & $f^{(2k+1)}(0) = (-1)^k$.

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x).$$

Now, applying the Remainder Estimation Theorem:

$$|R_{2k+1}(x)| = \left| \frac{f^{(2k+2)}(c)\, x^{2k+2}}{(2k+2)!} \right| , \quad c \in (0, x)$$

$$\leq \frac{1 \cdot |x|^{2k+2}}{(2k+2)!} \longrightarrow 0 \text{ as } k \longrightarrow \infty .$$

$\therefore \ |R_{2k+1}(x)| \longrightarrow 0$ as $k \to \infty$

Thus: $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} , \quad -\infty < x < \infty$

Example: Show that the Taylor Series for $f(x) = \cos x$ at $x = 0$ Converges to $\cos x$ for every $x$.

Sol: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + R_{2k}(x)$



$$|R_{2k}(x)| \leq \frac{1 \cdot |x|^{2k+1}}{(2k+1)!} \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

$$\Rightarrow \lim_{k \to \infty} |R_{2k}(x)| = 0. \Rightarrow \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} , \ -\infty < x < \infty.$$

$\therefore$

Example: Find the first four terms (non zero) in the Maclaurin series for :

$$\boxed{1} \quad \frac{1}{3}(2x + x\cos x) = \frac{2}{3}x + \frac{1}{3}x\cos x$$

$$= \frac{2}{3}x + \frac{1}{3}x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= \frac{2}{3}x + \frac{1}{3}x - \frac{1}{6}x^3 + \frac{1}{3\cdot 4!}x^5 - \frac{1}{3.6!}x^7 + \cdots$$

$$= 1x - \frac{1}{6}x^3 + \frac{1}{72}x^5 - \frac{1}{2160}x^7 + \cdots$$

$$\boxed{2} \quad e^x\cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= \left(1 + x + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(\cancel{\frac{x^2}{2!}} + \frac{x^3}{2!} + \frac{x^4}{2!\cdot 2!} + \frac{x^5}{2!\cdot 3!} + \cdots\right)$$

$$+ \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!\cdot 4!} + \frac{x^7}{3!\cdot 4!} + \cdots\right) + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

$\boxed{3}$ $\cos 2x$ :

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\Rightarrow \cos(2x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$$

Example: For what values of $x$ can we replace $\sin x$ by $x - \frac{x^3}{3!}$ with an error of magnitude Less than $3 \times 10^{-4}$ ?

Sol: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Using the Alternating Series Estimation Theorem; the error after $\frac{x^3}{3!} < \left|\frac{x^5}{5!}\right|$. Therefore, the error will be less than $3 \times 10^{-4}$ iff



$$\frac{|x|^5}{120} < 3 \times 10^{-4} \iff |x|^5 < 120 \times 3 \times 10^{-4}$$

$$\Rightarrow |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514.$$

**Example:** Estimate $\tan^{-1}(0.1)$ using $P_3(x)$, then estimate the error.

**sol:** $\tan^{-1}(0.1) = (0.1) - \frac{(0.1)^3}{3} = \frac{1}{10} - \frac{1}{3000} = \frac{299}{3000}$

$$|E| < \left|\frac{x^5}{5}\right| = \frac{|x|^5}{5} = \frac{(0.1)^5}{5} = 2\times 10^{-6}$$

Recall: $\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

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**Example:** Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$

**sol:** $f(x) = x - \frac{x^3}{3!} + R_3(x)$, where

$$R_3(x) = \frac{f^{(4)}(c)\, x^4}{4!}$$



Apply the Remainder Estimation theorem with $M = 1$



then $|R_3(x)| \leq \frac{|x^4|}{4!} = \frac{(0.1)^4}{4!} \approx 4.2\times 10^{-6}$

---

**Example:** Find Maclaurin Series for $f(x) = x^3 e^x$

**Sol:** $f(x) = x^3 e^x = x^3 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!}, \ \forall x.$

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**Example:** Find the Sum $S = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi)^{2n}}{2n!}$

**Sol:** $S = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (\pi^2)^n}{n!}$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-\pi^2)^n}{n!} = \frac{1}{2} e^{-\pi^2}$$

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**Example:** Find the Sum $S = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}$

$$= \cos \pi = \boxed{-1}.$$

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**Example:** Let $e^x \approx 1 + x + \frac{x^2}{2!}$ and $|x| < 0.1$

Estimate the error.



**Sol:** $R_2(x) = \frac{f'''(c)}{3!} x^3 \quad , \quad c \in (0, x)$

$$|R_2(x)| = \frac{e^c}{3!} x^3 < \frac{e^{0.1}}{3!} (0.1)^3 \approx 1.84 \times 10^{-4}.$$

e is Increasing.

## Lecture's Problems:

**Q22)** Use power series operations to find the Taylor series at $x=0$ for $f(x) = \frac{2}{(1-x)^3}$

**Sol:** Notice that $\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right)$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$\Rightarrow \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = 2 + 6x + 12x^2 + \cdots = \sum_{n=\boxed{2}}^{\infty} n(n-1)x^{n-2}$$

$$\Rightarrow \frac{2}{(1-x)^3} = \sum_{n=\boxed{0}}^{\infty} (n+2)(n+1)x^n.$$

**Q40)** The estimate $\sqrt{1+x} = 1 + \frac{x}{2}$ is used when $x$ is small. Estimate the error when $|x| < 0.01$.



**Sol:** $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$

By Alternating Series Estimation Theorem:

$$|\text{error}| < \left|\frac{-x^2}{8}\right| < \frac{(0.01)^2}{8} = 1.25 \times 10^{-5}.$$

# 10.10 The Binomial Series And Applications of Taylor Series:

**Def:** The Binomial Series:

For $-1 < x < 1$, $m \in \mathbb{R}$:

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k, \quad \text{where}$$

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!}, \text{ and}$$

The Combination: $$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}, \quad k \geq 3$$

**Example:** $$\binom{2}{1} = 2, \quad \binom{5}{2} = \frac{5(4)}{2!} = 10.$$

$$\binom{-1}{1} = -1, \quad \binom{-1}{2} = \frac{(-1)(-2)}{2!} = 1.$$




$$\binom{-1}{k} = \frac{(-1)(-2)(-3)\cdots(-1-k+1)}{k!} = \frac{(-1)(-2)(-3)\cdots(-k)}{k!}$$

$$= (-1)^k \cdot \frac{k!}{k!} = (-1)^k.$$

Corollary: $(1+ax)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} (ax)^k$

for $|ax| < 1$.

Example: $(1+x)^{-1} = 1 + \sum_{k=1}^{\infty} \binom{-1}{k} x^k$

$= 1 + \sum_{k=1}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$

Example: Find the first four non Zero terms for the binomial series.

$\boxed{1}$ $f(x) = \frac{1}{\sqrt[3]{1+x}} = (1+x)^{-\frac{1}{3}} = 1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{3}}{k} x^k$

$= 1 + \binom{-\frac{1}{3}}{1} x + \binom{-\frac{1}{3}}{2} x^2 + \binom{-\frac{1}{3}}{3} x^3 + \cdots$

$= 1 - \frac{1}{3}x + \frac{-\frac{1}{3}(-\frac{1}{3}-1)}{2!} x^2 + \frac{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)}{3!} x^3 + \cdots$



$= 1 - \frac{1}{3}x + \frac{(-\frac{1}{3})(-\frac{4}{3})x^2}{2} + \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{6} x^3 + \cdots$

$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots$

$\boxed{2}$ $$g(x) = \frac{x^3}{\sqrt[3]{1+x}} = x^3 (1+x)^{-\frac{1}{3}}$$

$$= x^3\left(1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots\right)$$

$$= x^3 - \frac{1}{3}x^4 + \frac{2}{9}x^5 - \frac{14}{81}x^6 + \cdots$$

$\boxed{3}$ $$h(x) = \frac{1}{\left(1-\frac{x}{2}\right)^2} = \left(1 - \frac{x}{2}\right)^{-2}$$

$$= 1 + \sum_{k=1}^{\infty} \binom{-2}{k}\left(\frac{-x}{2}\right)^k$$

$$= 1 + \binom{-2}{1}\left(\frac{-x}{2}\right)^1 + \binom{-2}{2}\left(\frac{-x}{2}\right)^2 + \binom{-2}{3}\left(\frac{-x}{2}\right)^3 + \cdots$$

$$= 1 + (-2)\left(-\frac{x}{2}\right) + \frac{(-2)(-3)\left(-\frac{x}{2}\right)^2}{2} + \frac{(-2)(-3)(-4)\left(\frac{-x}{2}\right)^3}{6} + \cdots$$

$$= 1 + x + \frac{3}{4}x^2 + \frac{x^3}{2} + \cdots$$

Remark: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

whenever $k \leq n$, and which is zero when $k > n$.

# Evaluating Nonelementry Integrals:

**Example:** Express $\int \sin(x^2)\,dx$ as a power series.

**Recall:** $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} \cdots$

$$\Rightarrow \int \sin(x^2)\,dx = \frac{x^3}{3} - \frac{x^7}{7.3!} + \frac{x^{11}}{11.5!} - \frac{x^{15}}{15.7!} + \cdots + C.$$

**Example:** Estimate $\int_0^1 \sin(x^2)\,dx$ with $|\text{error}| < 0.001$.

$$\int_0^1 \sin(x^2)\,dx = \frac{1}{3} - \frac{1}{7.3!} + \frac{1}{11.5!} - \frac{1}{15.7!} + \cdots \text{ (A.S)}$$

Notice that $\frac{1}{11.5!} \approx 0.00076 < 0.001$.

$$\Rightarrow \int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.310.$$



**Example:** Estimate $\int_0^1 \tan^{-1}x\,dx$ with $|\text{error}| < 0.02$

**Recall:** $\int_0^1 \tan^{-1}x\,dx = \int_0^1 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)dx$

$$= \left.\frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \cdots\right|_0^1$$

$$= \frac{1}{2} - \frac{1}{12} + \frac{1}{30} - \frac{1}{56} + \cdots$$

$$\Rightarrow \int_0^1 \tan^{-1} x \, dx \approx \frac{1}{2} - \frac{1}{12} + \frac{1}{30} \approx 0.45$$

since $\frac{1}{56} \approx 0.01786 < 0.02$.

(17) **Example:** Estimate $\displaystyle\int_0^{0.1} \frac{dx}{\sqrt{1+x^4}}$ with $|\text{error}| < 10^{-3}$.

$$\int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} = \int_0^{0.1} (1+x^4)^{-\frac{1}{2}} dx = \int_0^{0.1} \left(1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} x^{4k}\right) dx$$

$$= \int_0^{0.1} \left(1 - \frac{x^4}{2} + \frac{3x^8}{8} - \cdots\right) dx$$

$$= \left[ x - \frac{x^5}{10} + \frac{3x^9}{72} - \cdots \right]\Bigg|_0^{0.1}$$

$$= (0.1) - \frac{(0.1)^5}{10} + \frac{3(0.1)^9}{72} - \cdots$$



$$\Rightarrow \int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} \approx 0.1$$

since $|\text{Error}| < \frac{(0.1)^5}{10} \approx 0.000001 < 10^{-3}$.

# Evaluating Indeterminate Forms: $\left(\frac{0}{0}\right), \left(\frac{\infty}{\infty}\right), (\infty - \infty), (0 \cdot \infty)$

**Example:** Use Series to evaluate the limits.

① $\lim\limits_{x \to 1} \dfrac{\ln x}{x-1}$

**Recall** (10.7): $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots + (-1)^{n-1}\dfrac{x^n}{n} + \cdots$

whenever $-1 < x < 1$.

Replace $x$ by $(x-1)$, we get:

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n + \cdots$$

$$\therefore \lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1}\left(1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} - \cdots\right) = \boxed{1}$$

② $\lim\limits_{x \to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right) = \lim\limits_{x \to 0}\left(\dfrac{x - \sin x}{x \sin x}\right)$

$$= \lim_{x \to 0}\left[\frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}\right]$$

$$= \lim_{x \to 0} \frac{x^3\left(\frac{1}{3!} - \frac{x^2}{5!} - \cdots\right)}{x^2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)} = \boxed{0}$$

(29)

(3) $$\lim_{x\to 0} \frac{e^{x}-(1+x)}{x^{2}} = \lim_{x\to 0} \frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots-(1+x)}{x^{2}}$$

$$= \lim_{x\to 0} \frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots = \boxed{\frac{1}{2}}.$$

(33)

(4) $$\lim_{y\to 0} \frac{y-\tan^{-1} y}{y^{3}} = \lim_{y\to 0} \frac{y-\left(y-\frac{y^3}{3}+\frac{y^5}{5}-\frac{y^7}{7}+\cdots\right)}{y^{3}}$$

$$= \lim_{y\to 0} \left(\frac{1}{3}+\frac{-y^2}{5}+\frac{y^4}{7}-\cdots\right) = \boxed{\frac{1}{3}}$$

Euler's Identity

Recall: $i=\sqrt{-1}$, $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i, \ldots$

and $$e^{x}=\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$$

$$\Rightarrow e^{i\theta}=\sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} = 1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\cdots$$

 

$$= \left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\right)$$

$$\Rightarrow e^{i\theta} = \cos\theta + i\sin\theta$$ Euler's Identity.

where $\theta$ is a polar Angle. (real number).

Example: $e^{2+\frac{\pi}{3}i} = e^{2} e^{\frac{\pi}{3}i}$

$$= e^{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= e^{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{e^{2}}{2} + \frac{\sqrt{3}\,e^{2}}{2}i.$$

Example: $e^{i\pi} = \cos\pi + i\sin\pi = \boxed{-1}$

See table (10.1) page (602)



"End of Chapter 10"

and so on, to simplify the result, we obtain

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos\theta + i\sin\theta.$$

This does not *prove* that $e^{i\theta} = \cos\theta + i\sin\theta$ because we have not yet defined what it means to raise $e$ to an imaginary power. Rather, it says how to define $e^{i\theta}$ to be consistent with other things we know.

**DEFINITION**

For any real number $\theta$, $e^{i\theta} = \cos\theta + i\sin\theta$. (4)

Equation (4), called **Euler's identity**, enables us to define $e^{a+bi}$ to be $e^a \cdot e^{bi}$ for any complex number $a + bi$. One consequence of the identity is the equation

$$e^{i\pi} = -1.$$

When written in the form $e^{i\pi} + 1 = 0$, this equation combines five of the most important constants in mathematics.

**TABLE 10.1** Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$



## Exercises 10.10

**Binomial Series**

Find the first four terms of the binomial series for the functions in Exercises 1–10.

**1.** $(1 + x)^{1/2}$ **2.** $(1 + x)^{1/3}$ **3.** $(1 - x)^{-1/2}$

**4.** $(1 - 2x)^{1/2}$ **5.** $\left(1 + \frac{x}{2}\right)^{-2}$ **6.** $\left(1 - \frac{x}{3}\right)^{4}$

**7.** $(1 + x^3)^{-1/2}$ **8.** $(1 + x^2)^{-1/3}$