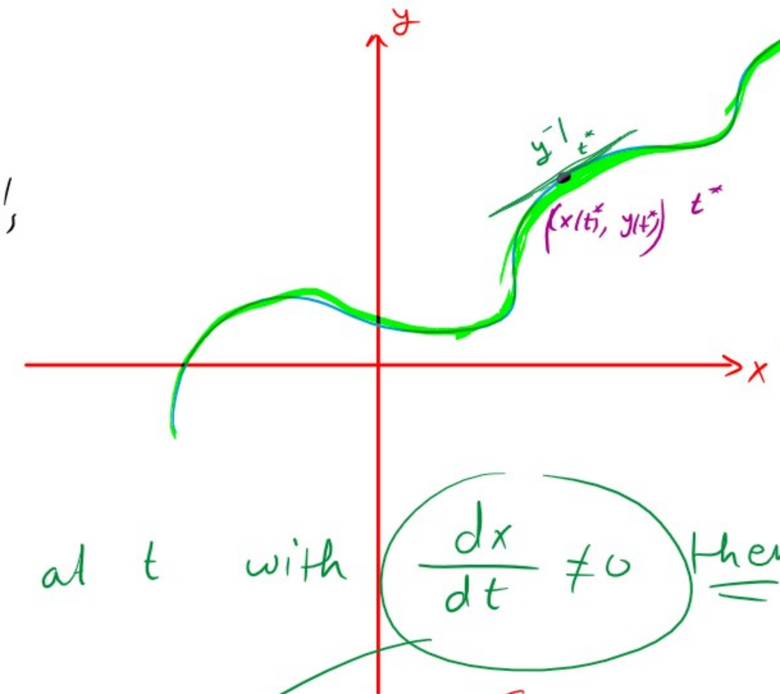
11.2

Wednesday, May 26, 2021 11:27 AM

11.1 Parametrization

$x = f(t)$, $y = g(t)$ } Parametric Eq's

$t \in I$ } Parameter Interval



Assume f, g, g' are diff at t with $\frac{dx}{dt} \neq 0$ then

① $y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

② $y'' = \frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$

Exp [9]: $x = 2t^2 + 3$, $y = t^4$ Find

(A) slope at $t = -1$

(B) tangent line at $t = -1$

(C) $\frac{d^2y}{dx^2}$ at $t = -1$

$x_0 = 2t^2 + 3 \big|_{t=-1} = 2 + 3 = 5$

$y_0 = t^4 \big|_{t=-1} = (-1)^4 = 1$

(A) $\frac{dy}{dx}\Big| = \frac{\frac{dy}{dt}}{} \Big| = \frac{4t^3}{} \Big| = \frac{-4}{-4} = 1$

Ⓐ $m = \text{slope} = \left.\dfrac{dy}{dx}\right|_{t=-1} = \left.\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=-1} = \left.\dfrac{\cancel{4t}}{\cancel{4t}}\right|_{t=-1} = \dfrac{-4}{-4} = 1$

Ⓑ $y - y_0 = m(x - x_0)$ $\quad x_0 = 5$ $\quad y_0 = 1$

$y = y_0 + \boxed{1}(x - x_0)$

$= 1 + x - 5$

$\boxed{y = x - 4}$

$\dfrac{dy}{dt} = y' = t^2$

$\dfrac{dy'}{dt} = 2t$

Ⓒ $\left.\dfrac{d^2y}{dx^2}\right|_{t=-1} = \dfrac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \left.\dfrac{2t}{4t}\right|_{t=-1} = \dfrac{-2}{-4} = \dfrac{1}{2}$

$\underline{\underline{Exp}}$ [15] Find the slope of the curve at $t = 2$ whose Parametric Eq's are $x^3 + 2t^2 = 9$, $2y^3 - 3t^2 = 4$

implicit differentiation

$\text{slope} = \left.\dfrac{dy}{dx}\right|_{t=2} = \left.\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=2}$

$3x^2x' + 4t = 0$

$\dfrac{dx}{dt} = x' = \dfrac{-4t}{3x^2}$

$6y^2y' - 6t = 0$

$y' = \dfrac{6t}{6y^2}$

$\dfrac{dy}{dt} = \dfrac{t}{y^2}$

$= \left.\dfrac{\frac{t}{y^2}}{\frac{-4t}{3x^2}}\right|_{t=2}$

$x^3 + 2t^2 = 9$

$= \dfrac{\frac{2}{2^2}}{\quad}$

$$= \frac{\frac{2}{2^2}}{\frac{-4(2)}{(3)\,1^2}}$$

$$= \frac{\frac{1}{2}}{-\frac{8}{3}}$$

$$= -\frac{3}{16}$$

$x^3 + 2t^2 = 9$

$x^3 + 2(2)^2 = 9$

$x^3 + 8 = 9$

$x^3 = 1$

$\boxed{x = 1}$

$2y^3 - 3t^2 = 4$

$2y^3 - 3(4) = 4$

$2y^3 - 12 = 4$

$2y^3 = 16$

$y^3 = 8$

$\boxed{y = 2}$

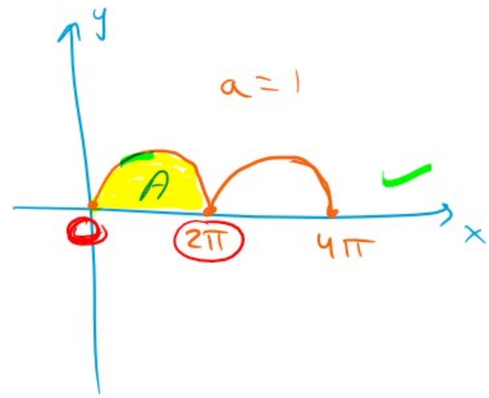
Exp Find the area under one arc of the cycloid:

$x = a(t - \sin t)$, $y = a(1 - \cos t)$ when $a = 1$

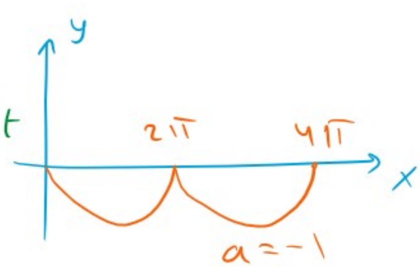
$a=1 \Rightarrow$ $\boxed{x = t - \sin t}$, $\boxed{y = 1 - \cos t}$

$dx = (1 - \cos t)\,dt$

$$A = \int_0^{2\pi} y\,dx \quad \text{or} \quad A = \int_0^{2\pi} x\,dy$$

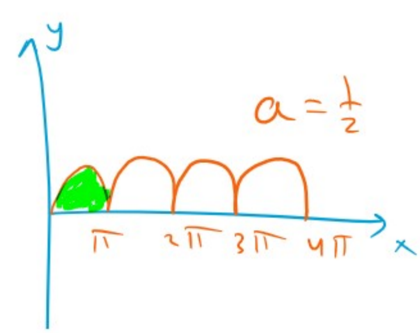


$$= \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\,dt = \int_0^{2\pi} (1 - \cos t)^2\,dt$$



$$= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt = \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2}\right)dt$$

$$= \left(t - 2\sin t + \frac{1}{2}t + \frac{\sin 2t}{4}\right)\Bigg|_0^{2\pi}$$



$$= \left(2\pi - \cancel{2\sin 2\pi} + \frac{1}{2}(2\pi) + \cancel{\frac{\sin 4\pi}{4}}\right) - \left(\cancel{0} - \cancel{2\sin 0} + \cancel{\frac{1}{2}(0)} + \cancel{\frac{\sin 0}{4}}\right)$$

$= 2\pi + \pi$

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$= 2\pi + \pi$

$= 3\pi$

$dx = -a \sin t \, dt$

(23) Exp: $x = a\cos t$, $y = b\sin t$, $\boxed{0 \le t \le 2\pi}$

Find area inside

$t=0 \Rightarrow IP \Rightarrow \left.\begin{matrix} x=a \\ y=0 \end{matrix}\right\} \Rightarrow (a,0)$

$t=2\pi \Rightarrow TP \Rightarrow \left.\begin{matrix} x=a \\ y=0 \end{matrix}\right\}$

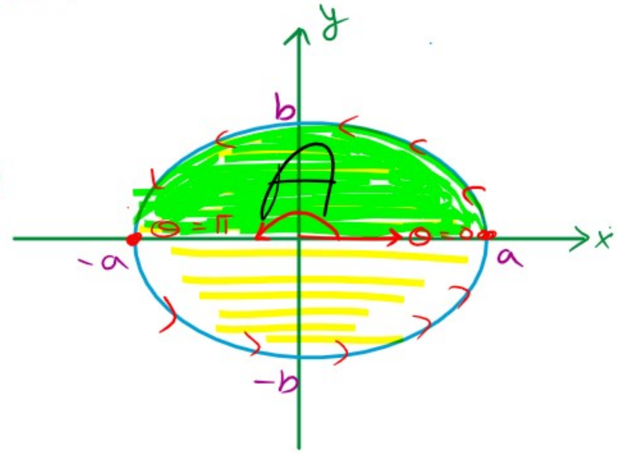
We need to find Cartesian Eq. (relation between x, y)

$$\frac{x}{a} = \cos t \quad , \quad \frac{y}{b} = \sin t$$

$$\frac{x^2}{a^2} = \cos^2 t \quad , \quad \frac{y^2}{b^2} = \sin^2 t$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow \text{Ellipse}$$



$x=0 \Rightarrow y^2 = b^2 \Rightarrow y = \pm b$

$y=0 \Rightarrow x^2 = a^2 \Rightarrow x = \pm a$

$$\text{Area} = 2A = 2\left|\int_0^{\pi} y\,dx\right| = 2\left|\int_0^{\pi} b\sin t\,(-a\sin t)\,dt\right|$$

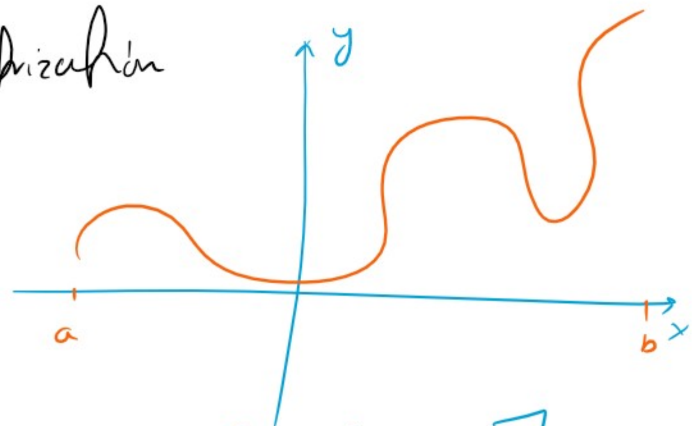
$$= 2ab\int_0^{\pi}\sin^2 t\,dt = \cancel{2}ab\int_0^{\pi}\frac{1-\cos 2t}{\cancel{2}}\,dt$$

$$= ab\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{\pi} = ab\left[\left(\pi - \cancel{\frac{\sin 2\pi}{2}}^{\,0}\right) - \left(\cancel{0} - \cancel{\frac{\sin 0}{2}}^{\,0}\right)\right]$$

$$= ab\pi \quad \checkmark$$

Arc length with Parametrization

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \\ t &\in [a,b] \end{aligned}\right\}\ \text{Parametrization}$$



$\frac{dx}{dt} = f'$

$\frac{dy}{dt} = g'$

Assume f', g' are cont. and not zero (both). Then length of this curve is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \quad \text{Parametrization}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2\left[1 + \frac{\left(\frac{dy}{\cancel{dt}}\right)^2}{\left(\frac{dx}{\cancel{dt}}\right)^2}\right]}\,dt$$

$$= \int_a^b \frac{dx}{dt}\sqrt{1 + \left(\frac{dy}{\ }\right)^2}\ \cancel{dt}$$

$$= \int_a \frac{dx}{\cancel{dt}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; \cancel{dt}$$

$f'(x) = \frac{dy}{dx}$

$$= \int_a^b \sqrt{1 + \left(f'(x)\right)^2} \, dx$$

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Exp 25 $\boxed{x = \cos t}$, $\boxed{y = t + \sin t}$, $\overset{a}{0} \le t \le \overset{b}{\pi}$

Find the length of this curve.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$\frac{dx}{dt} = -\sin t$

$\frac{dy}{dt} = 1 + \cos t$

$\left(\frac{dx}{dt}\right)^2 = \sin^2 t$

$\left(\frac{dy}{dt}\right)^2 = 1 + 2\cos t + \cos^2 t$

$$= \int_0^\pi \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t} \, dt$$

$$= \int_0^\pi \sqrt{2 + 2\cos t} \, dt$$

$$= \sqrt{2} \int_0^\pi \sqrt{1 + \cos t} \, dt$$

$$= \sqrt{2} \int_0^\pi \sqrt{(1 + \cos t) \frac{1 - \cos t}{1 - \cos t}} \, dt$$

$$= \sqrt{2}\int_0 \sqrt{(1+\cos t)\,\frac{1-\cos t}{1-\cos t}}\; dt$$

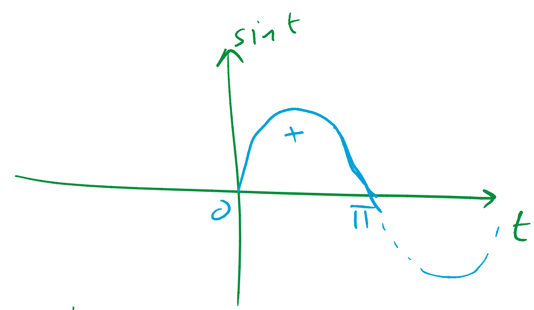
$$\sin^2 t + \cos^2 t = 1$$

$$\sin^2 t = 1 - \cos^2 t$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\frac{1-\cos^2 t}{1-\cos t}}\; dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\frac{\sin^2 t}{1-\cos t}}\; dt$$

$$\sqrt{\sin^2 t} = |\sin t| = \sin t$$



$$= \sqrt{2}\int_0^{\pi} \frac{\sin t \; dt}{\sqrt{1-\cos t}}$$

$$u = 1 - \cos t$$

$$du = \sin t \; dt$$

$$t = 0 \Rightarrow u = 0$$

$$t = \pi \Rightarrow u = 1 - -1 = 2$$

$$= \sqrt{2}\int_0^{2} \frac{du}{\sqrt{u}}$$

$$= \sqrt{2}\int_0^{2} u^{-\frac{1}{2}}\, du$$

$$= \sqrt{2}\left.\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right|_0^2 = 2\sqrt{2}\left.\sqrt{u}\right|_0^2 = 2\sqrt{2}\sqrt{2} - 0$$

$$= 2(2)$$

$$= \boxed{4}$$

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