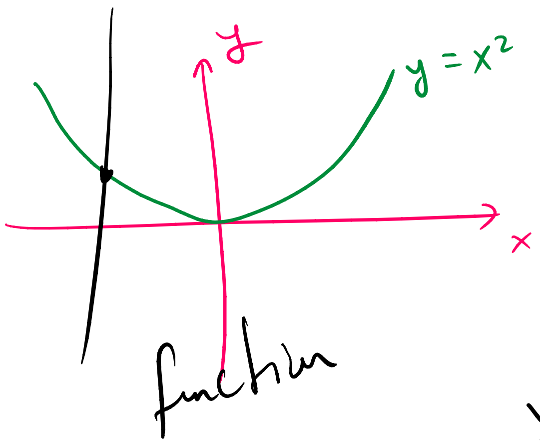
Inverse functions

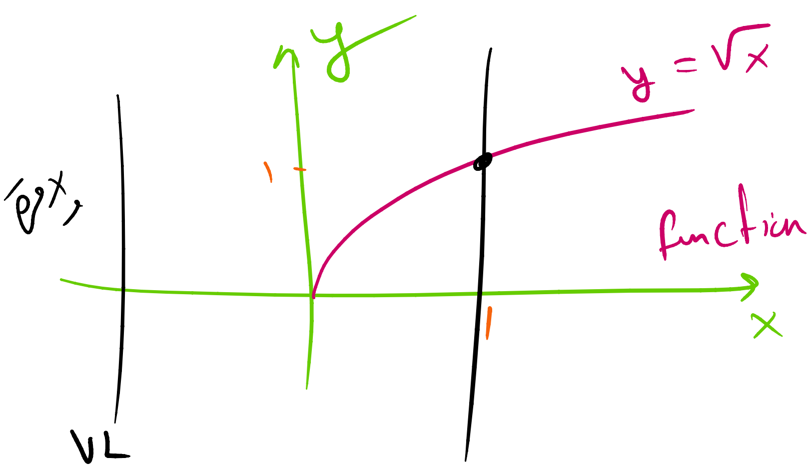


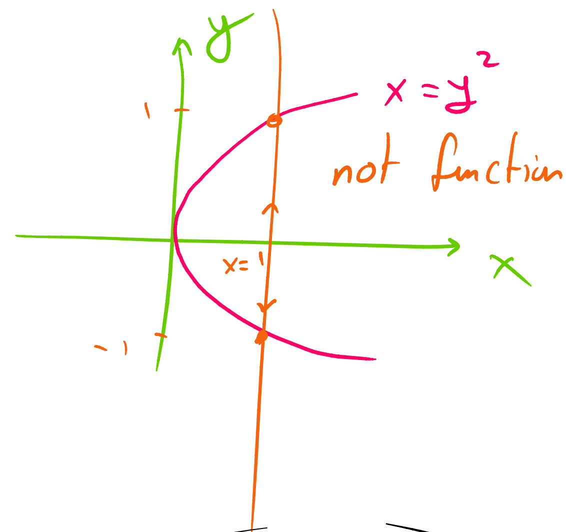
function: is relation "Rule" that assigns a unique value $y \in R$ to each element in domain $x \in D$

VLT
vertical line

if any VL crosses f at most once the f is function

e.g.,



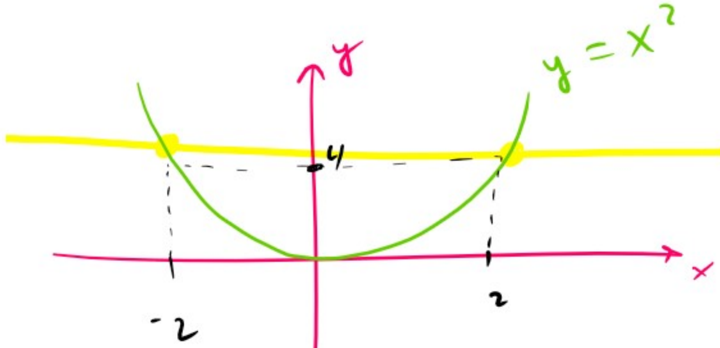


Q. what is 1-1 function?

one-to-one functions

A: f(x) is 1-1 function on domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2 \in D$

Exp



Is y 1-1?

$x_1 = 2$ $\quad x_1 \neq x_2$

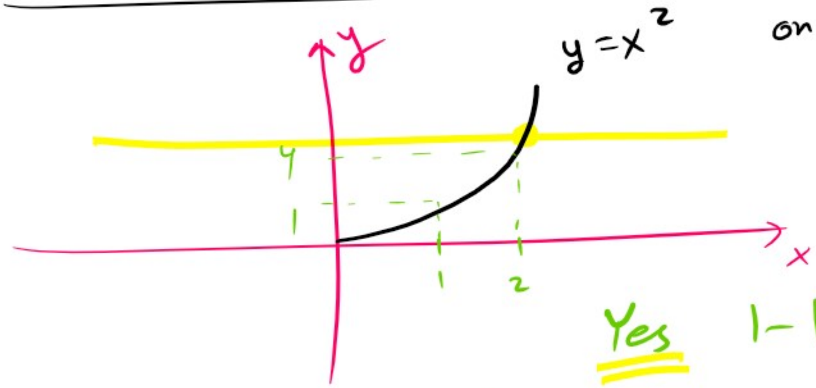
$x_2 = -2$

$f(x_1) = f(x_2) = 4$

not 1-1 function

$D = (-\infty, \infty)$

Exp



on $x \geq 0$

Is y 1-1?

Yes 1-1 fuction

$D = [0, \infty)$

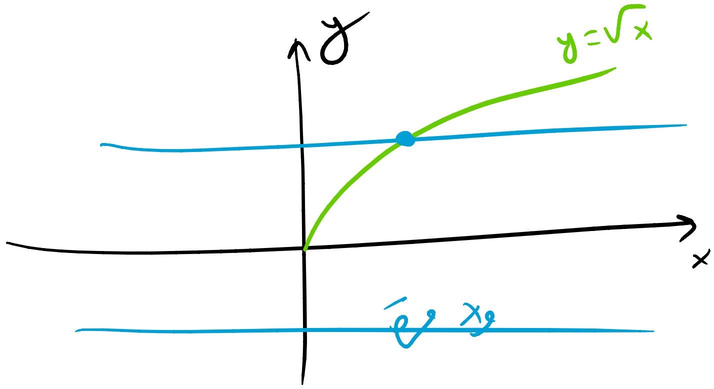
We use HLT to check if f is 1-1 or not

HLT: Horizontal Line Test

If any HL crosses f at most once, then f is 1-1 ✓

... $\sqrt{x}$ ... Is y 1-1?

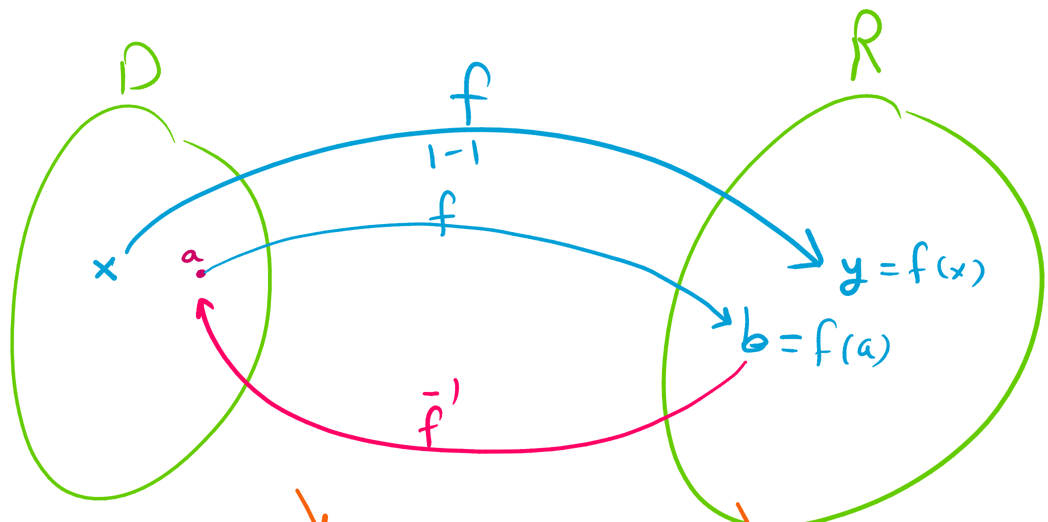
Exp $y = \sqrt{x}$ Is y 1-1?



1-1 ✓

Q: why we need 1-1 functions?

A:



$f : D \longrightarrow R$ | $f^{-1} : R \longrightarrow D$

$a \longrightarrow f(a) = b$ | $b \longrightarrow f^{-1}(b) = a$

$f^{-1}(f(a)) = f^{-1}(b)$

$a = f^{-1}(b)$

Domain f is Range of f^{-1}
Range f is Domain of f^{-1}

Inverse of f is denoted by $f^{-1}(x)$

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

$$[f(x)]^{-1} = \frac{1}{f(x)} \checkmark$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \quad \forall x \in D(f)$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x \quad \forall x \in D(f^{-1})$$

$D(f^{-1})$ ↓ $R(f)$

Q: Given ... function $f(x)$

Q: Given 1-1 function $f(x)$
How to find $f^{-1}(x)$?

A:
[1] Replace $f(x)$ by y
[2] Solve for x نضع x موضوع القانون
[3] Replace x by y
Replace y by x
[3] Replace y by $f^{-1}(x)$

Exp Given $f(x) = x^2$, $x \geq 0$
① Find $f^{-1}(x)$

[1] $y = x^2$

[2] $\sqrt{y} = \sqrt{x^2} \Rightarrow \sqrt{y} = |x| \Rightarrow x = \sqrt{y}$

③ $y = \sqrt{x}$

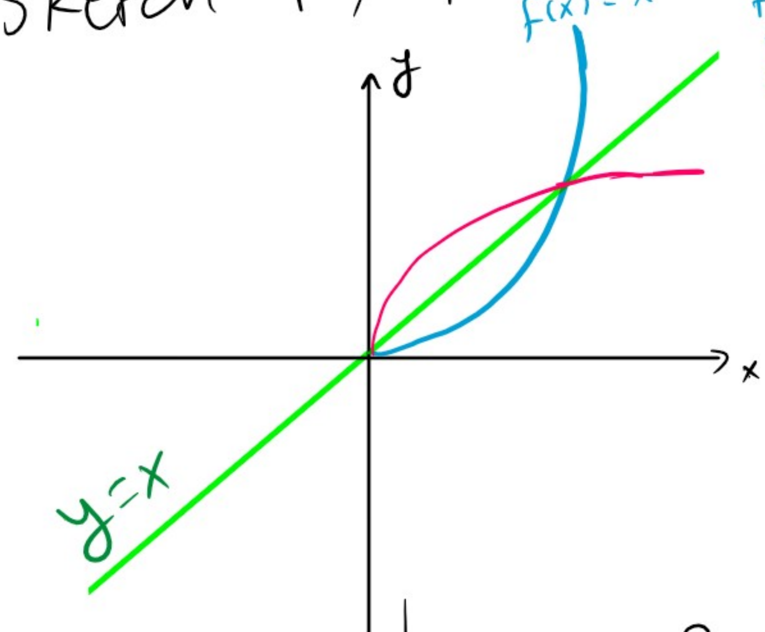
④ $f^{-1}(x) = \sqrt{x}$

② Find $D(f)$, $D(f^{-1})$, $R(f)$, $R(f^{-1})$

$D(f) = [0, \infty) = R(f^{-1})$

$D(f^{-1}) = [0, \infty) = R(f)$

③ Sketch f, f^{-1}



④ What symmetry we see? $\Rightarrow$ f, f^{-1} symmetric about $y = x$

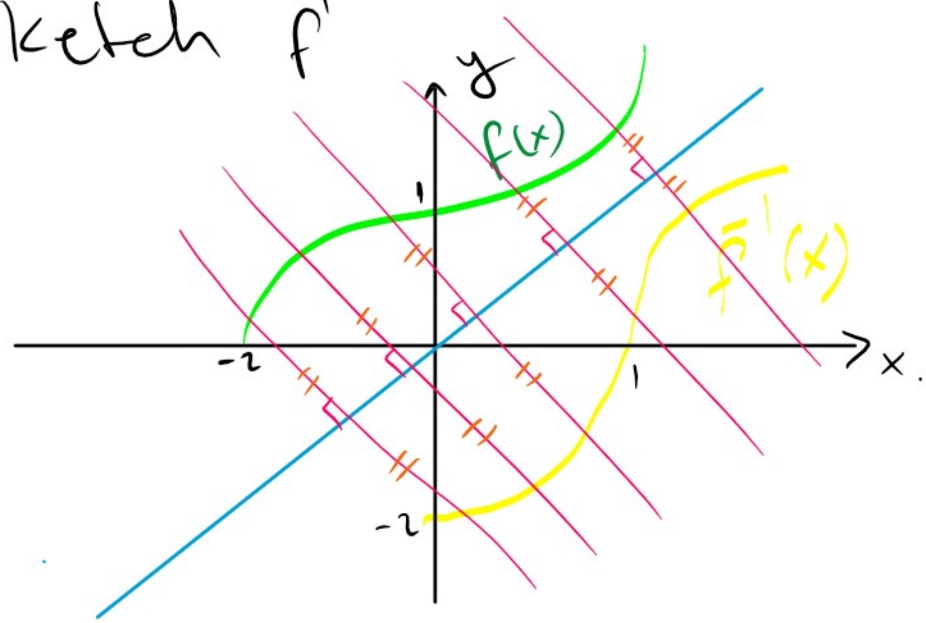
⑤ find $(f^{-1} \circ f)(x)$, $(f \circ f^{-1})(x)$

(5) find $(f^{-1} \circ f)(x)$, $(f \circ f^{-1})(x)$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x|$$
$$= x$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

Exp Give f in this graph
Sketch f^{-1}



Exp Find f^{-1} if $f(x) = x^2 - 2x$, $x \le 1$

$D(f) = (-\infty, 1] = R(f^{-1})$

(1) $y = x^2 - 2x$

(1) $y = x^2 - 2x$

(2) $y = \underline{x^2 - 2x + 1} - 1 = (x-1)^2 - 1$

$+1 \qquad\qquad +1$

$\sqrt{(x-1)^2} = \sqrt{y+1}$

$|x-1| = \sqrt{y+1}$

$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x \leq 1 \end{cases}$

$1 - x = \sqrt{y+1}$

$-1 \qquad -1$

$-x = -1 + \sqrt{y+1}$

$\boxed{x = 1 - \sqrt{y+1}}$

(3) $y = 1 - \sqrt{x+1}$

(4) $f^{-1}(x) = 1 - \sqrt{x+1}$

$x + 1 \geq 0$

$x \geq -1$

$D(f^{-1}) = [-1, \infty) = R(f)$

Q. How can we derive f^{-1}

A. See next result

Th $f : D \longrightarrow R$ is 1-1 and f' exists and never zero on D

$a \longrightarrow b = f(a)$

Then $f^{-1} : R \longrightarrow D$ is diff on R with

$b \longrightarrow a$

$b = f(a)$

$f^{-1}(b) = f^{-1}(f(a))$

$f^{-1}(b) = a$

$$\left.\frac{d f^{-1}}{dx}\right|_{x=b} = \frac{1}{f'(f^{-1}(b))}$$

$x = b \to f(a)$, $f^{-1}(b) \to a$

$f : D \longrightarrow R$, $a \longrightarrow b = f(a)$

$f' : D \longrightarrow R$

$f'' : D \longrightarrow R$

$\Rightarrow$ $f(a)$, $f'(a)$, $f''(a)$ $\quad a \in D(f)$

$f^{-1} : R \longrightarrow D$, $b \longrightarrow a = f^{-1}(b)$

$(f^{-1})' : R \longrightarrow D$

$f^{-1}(b)$, $(f^{-1})'(b)$ $\Rightarrow b \in D(f^{-1}) = R$

$(f^{-1})' : R \longrightarrow D$ ⟩ $(f^{-1})'(b)$

$(f^{-1})'' : R \longrightarrow D$ ⟩ $(f^{-1})''(b)$

Exp $y = x^2$, $x \geq 0$

Find $(f^{-1})'(y)$

$b = 4$

$f(a) = 4$ → $f(a) = b$

$a^2 = 4$

$a = \pm 2 \Rightarrow a = 2$

$f(x) = x^2$

$f'(x) = 2x$

S1

$$(f^{-1})'(4) = \frac{1}{f'(a)} = \frac{1}{f'(2)}$$

$a = f^{-1}(b)$

$$= \frac{1}{2(2)}$$

$$= \frac{1}{4}$$

S2 $f^{-1}(x) = \sqrt{x}$

$$(f^{-1})' = \frac{1}{2\sqrt{x}}$$

$$(f^{-1})'(4) = \frac{1}{\ldots} = \frac{1}{\ldots} = \frac{1}{4}$$

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}$$

Exp Let $f'(x) = 3x^2$

$f(a) = b$

Find $\left.\frac{df^{-1}}{dx}\right|_{x = f(\sqrt{2})}$

$f(a) = f(\sqrt{2})$

$a = \sqrt{2}$

$$\left.\frac{df^{-1}}{dx}\right|_{x = f(\sqrt{2}) \to b} = \frac{1}{f'(a)} = \frac{1}{f'(\sqrt{2})} = \frac{1}{3(\sqrt{2})^2}$$

$$= \frac{1}{3(2)} = \frac{1}{6}$$

Exp $f(x) = 5x + e^{2x}$

Find $(f^{-1})'(1)$

$f(a) = b$

$f(a) = 1$

$5a + e^{2a} = 1$

$a = ??$ 0

$$\left.\frac{df^{-1}}{dx}\right|_{x=1 \to b} = \frac{1}{f'(a)} = \frac{1}{f'(0)}$$

$f' = 5 + 2e^{2x}$

$$= \frac{1}{\;} = \frac{1}{5+2} = \frac{1}{7}$$

$$= \frac{1}{5+2e^0} = \frac{1}{5+2(1)} = \frac{1}{5+2} = \frac{1}{7}$$

⑫ Exp

$$y = f(x) = 1 - \frac{1}{x}, \quad x > 0$$

① find $R(f^{-1}) = D(f) = (0, \infty)$

② Find f^{-1}

[1] $y = 1 - \frac{1}{x}$

$$\frac{1}{x} = 1 - y$$

$$x = \frac{1}{1-y}$$

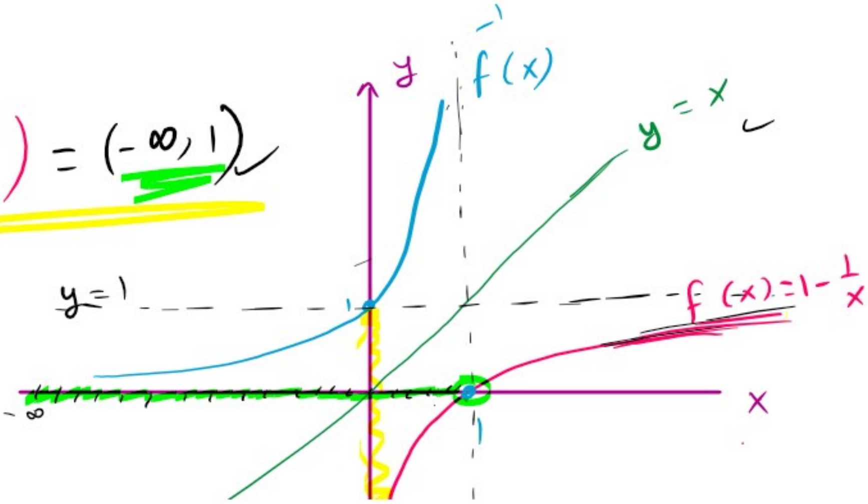
[2] $y = \frac{1}{1-x}$

[3] $f^{-1}(x) = \frac{1}{1-x}$ → قطعي → $D(f^{-1}) = \mathbb{R} \setminus \{1\}$

[3) Find $R(f)$

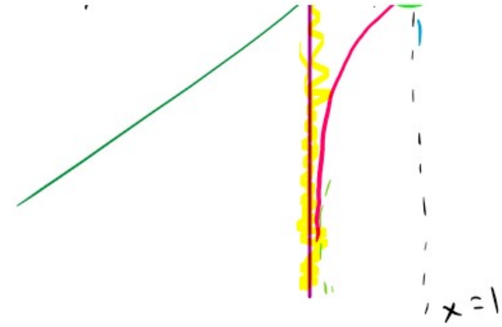
$$R(f) = D(f^{-1}) = (-\infty, 1)$$

$f(x) = 1 - \frac{1}{x}, \quad x > 0$



$f(x) = ①$ x

$f^{-1}(x) = \frac{1}{1-x}$ ✓



Q31 $f(x) = \frac{x+3}{x-2}$ → $D(f) = \mathbb{R} \setminus \{2\}$

1-1

① Find f^{-1}

① $y = \frac{x+3}{x-2}$

② $y(x-2) = x+3$

$yx - 2y = x + 3$

$yx - x = 2y + 3$

$x(y-1) = 2y + 3$

$x = \frac{2y+3}{y-1}$

③ $y = \frac{2x+3}{x-1}$

④ $f^{-1}(x) = \frac{2x+3}{x-1}$

② $D(f^{-1}) = \mathbb{R} \setminus \{1\} = R(f)$

③ $R(f^{-1}) = D(f) = \mathbb{R} \setminus \{2\}$

④ Show that $f(f^{-1}(x)) = x$

$f = \dfrac{x+3}{x-2}$

$$f\left(\frac{2x+3}{x-1}\right) = \frac{\frac{2x+3}{x-1}+3}{\frac{2x+3}{x-1}-2} = \frac{\frac{2x+3+3(x-1)}{x-1}}{\frac{2x+3-2(x-1)}{x-1}}$$

$$= \frac{2x+3+3x-3}{2x+3-2x+2} = \frac{5x}{5} = x$$

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(41) $f(x) = x^3 - 3x^2 - 1, \quad x \geq 2$

Find $\left.\dfrac{df^{-1}}{dx}\right|_{x=-1}$

$x = -1 = f(3)$ ($-1 \to b$, $3 \to a$)

$f' = 3x^2 - 6x$

$f'(3) = 3(3)^2 - 6(3)$

$= 3(9) - 18$

$= 27 - 18$

$= 9$

$$\left.\frac{df^{-1}}{dx}\right| = \frac{1}{\quad} = \frac{1}{\quad}$$

$$\frac{df}{dx}\Bigg|_{x=-1} = \frac{1}{f'(a)} = \frac{1}{f'(3)}\Bigg| = 9$$

$$= \frac{1}{9}$$ ✓

$$f(x) = 1 - \frac{1}{x} = \frac{x}{x} - \frac{1}{x} = \frac{x-1}{x}$$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \infty} \frac{1x-1}{1x} = 1 \Rightarrow y = 1 \text{ H. Asy}$$

$x = 0$ V. Asy since $\lim_{x \to 0^+} \frac{x-1}{x} = \frac{0-1}{\text{small } +} = -\infty$

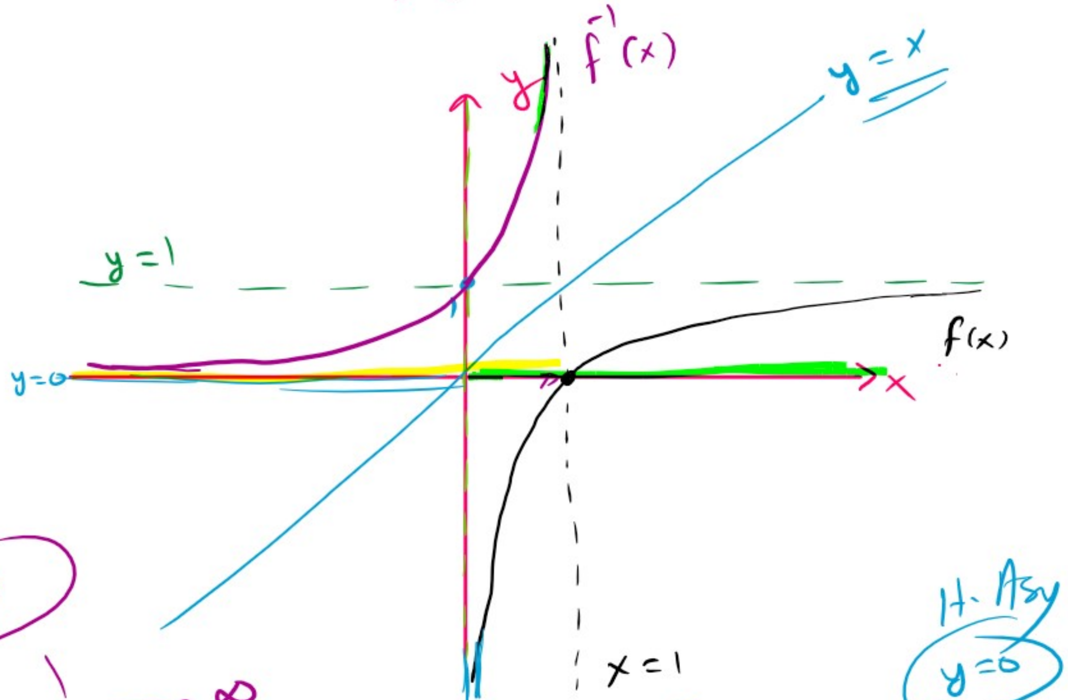
$$y = \frac{x-1}{x}$$

Keypoint (1,0)

$$f^{-1}(x) = \frac{1}{1-x}$$

$x = 1$ V. Asy ✓

$$\lim \frac{1}{\ldots} = \frac{1}{\text{small } +} = \infty$$



$\lim_{x \to 1^-} \frac{1}{1-x} = \frac{1}{\text{small} +} = \infty$

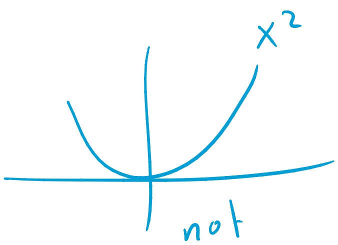
$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{1-x} = 0$

$f(x) = x^2$ on $x \geq 0$

$f(x) = x^2$ on $x \geq 2$

(1)
(2) $D(f) = R(f^{-1})$
(3) $D(f^{-1}) = R(f)$
(4)

Only 1-1 functions have inverse

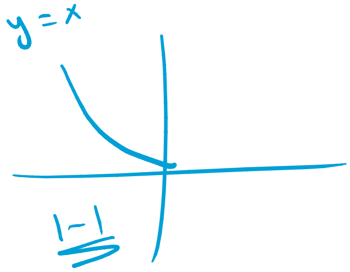


no inverse

$\Rightarrow$

$x \leq 0$

$y = x^2$



it has inverse

$D(f) = R(f^{-1})$

$D(f^{-1}) = R(f)$