Integral Test (Conv./Div) IT

$\sum_{n=1}^{\infty} a_n$??

Consider $\sum_{n=k}^{\infty} a_n$, where

- a_n positive terms and
- $a_n = f(n)$ is cont., +, ↓ on $[k, \infty)$

Then $\sum_{n=k}^{\infty} a_n$ and $\int_k^{\infty} f(x)\,dx$ both conv. or both div

Exp Does the following Series Conv. / Div ?

① $\sum_{n=1}^{\infty} \frac{1}{n^2}$

- $a_n = \frac{1}{n^2}$ positive term $\forall\, n = 1,2,3,\dots$
- $f(x) = \frac{1}{x^2}$ is +, cont., ↓ on $[1,\infty)$

$\int_1^{\infty} \frac{1}{x^2}\,dx = \frac{1}{\boxed{2} - 1} = \frac{1}{1} = 1$

Hence $\sum_{n=1}^{\infty} \frac{1}{n^2}$ con ~~to 1~~ by IT

② $\sum_{n=1}^{\infty} \frac{1}{n^3}$ Con by IT

conv

Exp^* $\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \le 1 \end{cases}$ $\frac{1}{p-1}$: conv, ∞: Div ✓

Exp^{**} $\int_0^1 \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p \ge 1 \end{cases}$ $\frac{1}{1-p}$: conv, ∞: div

Test P-series Test :

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv if $p > 1$ by IT by Exp^*

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ div if $p \le 1$

③ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ div p-series $p = \frac{1}{2}$

$\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} a_n = 0$

$\sum^{\infty}$ 1 • $a_n = \frac{1}{n^2+1}$ positive terms $\forall\, n = 1, 2, \ldots$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\quad \lim_{n\to\infty} a_n = 0$

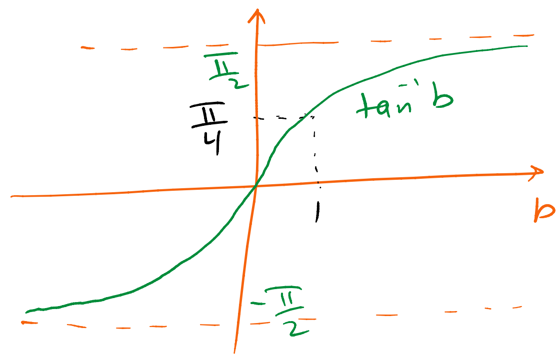
- $a_n = \dfrac{1}{n^2+1}$ positive terms $\forall\, n = 1, 2, \ldots$
- $f(x) = \dfrac{1}{x^2+1}$ is $+$, cont, $\downarrow$ on $[1,\infty)$

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1} x \Big|_1^b$$

$$= \lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} 1\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$



Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ Conv by IT

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$ $\quad \lim_{n\to\infty} a_n = 0$

- $a_n = \dfrac{1}{2n-1}$ is positive for all $n = 1, 2, 3 \ldots$
- $f(x) = \dfrac{1}{2x-1}$ is $\downarrow$, $+$, cont. on $[1,\infty)$

$x = \frac{1}{2} \notin [1,\infty)$

$$\int_1^{\infty} \frac{1}{2x-1}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{2x-1}\, dx$$

$u = 2x-1$

$du = 2\, dx$

$\frac{du}{2} = dx$

$x = 1 \Rightarrow u = 1$

$x = b \Rightarrow u = 2b-1$

$$= \lim \ \frac{1}{2} \int \frac{du}{}$$

$$= \lim_{b\to\infty} \frac{1}{2}\int \frac{du}{u}$$

$x = b \Rightarrow u = 2b-1$

$$= \lim_{b\to\infty} \frac{1}{2} \ln|u| \Big|_{1}^{b} = \frac{1}{2} \lim_{b\to\infty} \ln|2x-1| \Big|_{1}^{b}$$

$$= \frac{1}{2} \lim_{b\to\infty} \Big[\ln(2b-1) - \underbrace{\ln 1}_{0}\Big]$$

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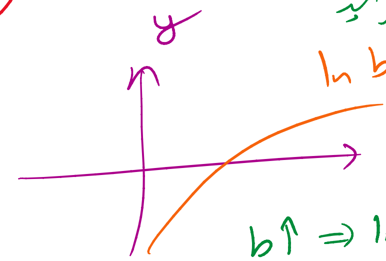

$b\uparrow \Rightarrow \ln b \uparrow$

$$= \frac{1}{2} \lim_{b\to\infty} \ln(2b-1)$$

ln cont.

$$= \frac{1}{2} \ln \lim_{b\to\infty} (2b-1) \to \infty$$

$$= \frac{1}{2} \ln \infty$$

$$= \infty$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ div by IT

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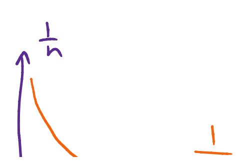
6 $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$

$u = \frac{1}{n}$

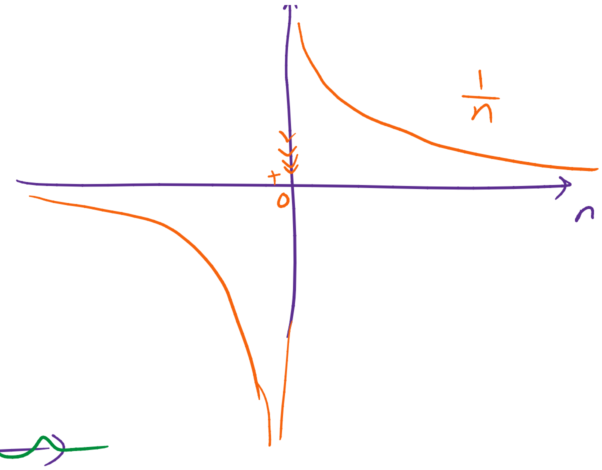
$n \to \infty \Rightarrow u \to 0^+$

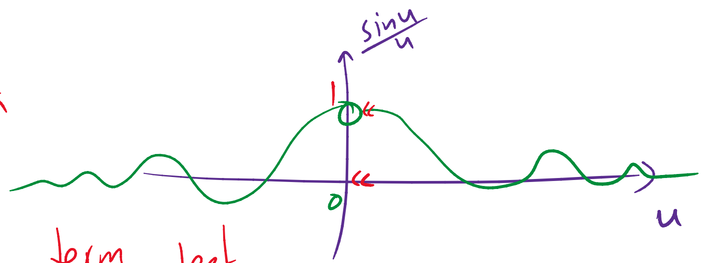
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$$



$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dots$ $\quad n \to$

$$= \lim_{u\to 0^+} \frac{\sin u}{u} = 1 \neq 0$$





Hence, $\sum n \sin \frac{1}{n}$ div by n^{th} term test

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(26) $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(\sqrt{n}+1)}$$

- $a_n = \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ postive terms $\forall\ n = 1, 2, \dots$
- $f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$ is $+$, $\downarrow$, cont on $[1, \infty)$

$$\int_1^{\infty} \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$

$u = \sqrt{x} + 1$

$du = \frac{1}{2\sqrt{x}}\, dx$

$2\,du = \frac{dx}{\sqrt{x}}$

$$= \lim_{b\to\infty} \int \frac{2\,du}{u}$$

$$= \lim_{b\to\infty} 2 \ln|u| \Big|_{\boxed{1}}^{\boxed{b}} = 2 \lim_{b\to\infty} \ln|\sqrt{x}+1| \Big|_{\boxed{1}}^{b}$$

$$= 2 \lim_{b\to\infty} \Big[\underbrace{\ln(b+1)}_{\infty} - \underbrace{\ln 2}_{-\ln 2} \Big]$$

Hence, $\sum^{\infty} \frac{1}{\dots}$ div by IT

Hence, $\sum \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ div by $\perp$ '

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(34) $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$ Apply n^{th} term Test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \tan \frac{1}{n} = \lim_{n\to\infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}} \quad \left(\frac{1}{n} \to u\right)$$

$$= \lim_{u\to 0^+} \frac{\tan u}{u} \quad \frac{0}{0} \quad \longrightarrow 1$$

$$= \lim_{u\to 0^+} \frac{\sec^2 u}{1} = \sec^2 0 = \frac{1}{\cos^2 0}$$

$$= \frac{1}{1^2} = \underline{\underline{1}} \neq 0$$

Hence, $\sum n \tan \frac{1}{n}$ div by n^{th} term test.

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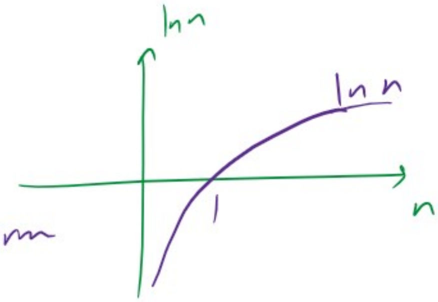
(8) Use $\underline{\underline{IT}}$ to check Conv / Div

$$\sum_{n=2}^{\infty} \frac{\ln n^2}{n}$$



$a_n = \frac{\ln n^2}{n}$ postive term

$\underline{\underline{f(x)}} = \frac{\ln x^2}{x}$ positive on $[2, \infty)$

cont. on $[2, \infty)$

$\boxed{\downarrow}$

$$f'(x) = \frac{x\left(2\frac{1}{x}\right) - \ln x^2}{x^2} = \frac{2 - \ln x^2}{x^2}$$

$f(x) = \frac{\quad}{x^2}$ $\quad x^2 \nearrow +$

$f'(e^2) = \frac{2 - 2\ln e^2}{(+)} < 0$

$2 - \ln x^2 = 0$

$2 = \ln x^2$

$2 = 2\ln x$

$1 = \ln x$

$e^1 = e^{\ln x}$

$\boxed{e = x}$

f' $+ + + + \; | \; - - - - -$

e $\quad e^2$

$\approx 2.718\ldots$

$$\sum_{n=2}^{\infty} \frac{\ln n^2}{n} = \frac{\ln 2^{\cancel{2}}}{\cancel{2}} + \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$$

$n=2$

$n=2$

$b_n \; +$

$f(x) = \frac{\ln x^2}{x}$ $+, \downarrow$, cont on $[3, \infty)$

$$= \ln 2 + \sum_{n=3}^{\infty} \frac{\ln n^2}{n} = \boxed{\infty}$$

$$\int_3^{\infty} f(x)\,dx = \int_3^{\infty} \frac{\ln x^2}{x}\,dx = \lim_{b\to\infty} \int_3^{b} 2\frac{\ln x}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\lim_{b\to\infty} \int_3^{b} 2u\,du = \lim_{b\to\infty} u^2 \Big|_3^b$$

$$= \lim_{b\to\infty} (\ln x)^2 \Big|_3^b$$

$b \to \infty \qquad 3$

$$= \lim_{b \to \infty} \left(\underset{\infty}{(\ln b)^2} - \underset{\varepsilon}{(\ln 3)^2} \right)$$

$$= \infty$$

Hence $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n^2}{n}$ div by IT

Hence, $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n^2}{n} = \ln 2 + \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$ div

$$= \ln 2 + \infty$$

$$= \infty$$