

Chapter 5. Series Solutions of second Order Linear Eq

5.1

Review of power series

In this chapter, we discuss the use of power series to construct a fundamental sets of solutions y_1 and y_2 of 2nd order linear D.Es whose coefficients are functions of the Independent variable and we write the solutions y_1 and y_2 in term of power series.

Recall: Some results about power series that we need.

(1) A power series about the point x_0 "center" has

the form: $\sum_{n=0}^{\infty} a_n (x - x_0)^n$. And it is

said to converge at x if

$\lim_{k \rightarrow \infty} \sum_{n=0}^k a_n (x - x_0)^n$ exists for that x .

(2) The series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is said to Converge

absolutely at a point x , if the series

$$\sum_{n=0}^{\infty} |a_n (x-x_0)^n| \text{ Converges.}$$

(3) To test the absolute convergence for the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$, we use the ratio test.

$$\text{Ratio test: } \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1} (x-x_0)^{n+1}}{a_n (x-x_0)^n} \right|$$

$$= |x-x_0| \left(\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \right) = L$$

$$= |x-x_0| \cdot L,$$

then the power series converges absolutely if

① $|x-x_0| \cdot L < 1$.

② And diverges if $|x-x_0| \cdot L > 1$

③ And if $|x-x_0| \cdot L = 1$, then the test is inconclusive.

Example: For which values of x does the power

series $\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n$ converge?

Sol: $\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} (n+1) (x-2)^{n+1}}{(-1)^{n+1} n (x-2)^n} \right|$

$$= |x-2| \lim_{n \rightarrow \infty} \frac{n+1}{n}$$

$$= |x-2| \cdot 1 < 1$$

$$\Rightarrow -1 < x-2 < 1 \iff 1 < x < 3$$

At $x=1$, then: $\sum_{n=1}^{\infty} (-1)^{n+1} n (-1)^n = \sum_{n=1}^{\infty} -n$
which is diverge. nth term test

At $x=3$, then: $\sum_{n=1}^{\infty} (-1)^{n+1} n (1)^n = \sum_{n=1}^{\infty} (-1)^{n+1} n$

which is diverge, since $\lim_{n \rightarrow \infty} (-1)^{n+1} n \neq 0$
nth term test Alternating

Then the interval of absolute convergence is $(1, 3)$.

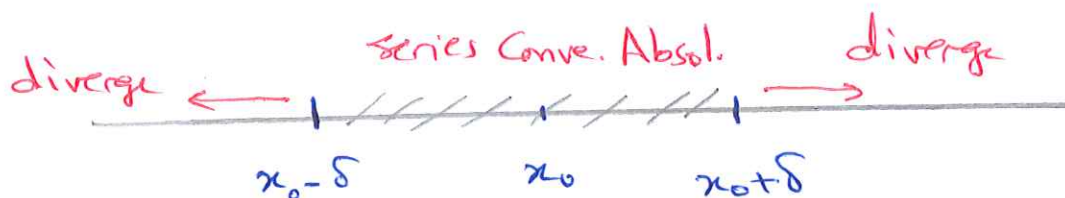
4) The radius of Convergence: is a positive

number ρ such that $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

converges absolutely for $|x-x_0| < \rho$, and

diverges for $|x-x_0| > \rho$.

The Interval $|x-x_0| < \rho$ is called the Interval of Convergence



At $x_0 - \delta$ & $x_0 + \delta$, the series may converge or diverge

Example: Determine the radius of Convergence of the

power series $\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n$

So li $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} |x-2| \left(\frac{n+1}{n} \right) = |x-2| < 1$

previous example

center ρ

\Rightarrow The radius of Convergence = $\rho = 1$.

5) Differentiation and Integration of power series

If $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, then

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1} = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$f''(x) = \sum_{n=0}^{\infty} n(n-1) a_n (x-x_0)^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

$$\text{And } \int f(x) dx = \sum_{n=0}^{\infty} \frac{a_n (x-x_0)^{n+1}}{n+1} + C$$

6) The Taylor series for the function f about $x=x_0$

$$\text{is } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n.$$

Note: A function f that has a Taylor series expansion

about $x=x_0$ with radius of convergence $r>0$

is said to be analytic at $x=x_0$,

Like: $\sin x$, $\cos x$, e^x , ...

7) Shifting of Index of Summation.

Example: Write the series $\sum_{n=2}^{\infty} (n+2)(n+1) a_n (x-1)^{n-1}$

as a series involves $(x-1)^n$.

Sol: Let $m = n - 2 \Rightarrow n = m + 2$

When $n = 2$, then $m = 0$

$$\therefore \sum_{m=0}^{\infty} (m+4)(m+3) a_{m+2} (x-1)^m$$

(change m by n)

Example: Write the given expressions as a single sum involves x^n (Generic term involves x^n)

$$(1) \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2 a_n x^n$$

$$= \sum_{n=0}^{\infty} [(n+1) a_{n+1} + 2 a_n] x^n$$

$$\begin{aligned}
 (2) \quad & x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \\
 \Rightarrow & \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\
 = & \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n \\
 = & \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n \\
 = & a_0 + \sum_{n=1}^{\infty} [(n+1)(n) a_{n+1} + a_n] x^n.
 \end{aligned}$$

$$8) \text{ If } \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

for all x in some open Interval with center x_0

then $a_n = b_n, \forall n = 0, 1, 2, \dots$

In particular, if $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0, \forall x$

then $a_0 = a_1 = \dots = a_n = \dots = 0$.

page 253. Rewrite the given expression as a sum

whose generic term includes x^n .

$$(24) \quad (1-x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\text{sol:} \quad \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} [(n+2)(n+1) a_{n+2} - n(n-1) a_n] x^n$$

$$(27) \quad x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

$$\text{sol:} \quad \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + \sum_{n=1}^{\infty} [a_n + n(n+1) a_{n+1}] x^n$$

5.2 series solutions Near an ordinary points.

Part I:

In ch 3, we described methods of solving second order linear D.Es with constant coefficients.

We now consider methods of solving second order linear D.Es when the coefficients are functions of the independent variable t . (or x)

It is sufficient to consider the homogeneous eq.

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad \dots (1)$$

since the procedure for the nonhomogeneous Eq. is similar.

Def: A point x_0 such that $P(x_0) \neq 0$ in Eq. (1) is called an ordinary point.

If $P(x_0) = 0$, then x_0 is called Singular Point.

We will assume that P , Q and R in Eq. (1) are continuous. It follows that there is an Interval about x_0 in which $P(x)$ is never zero.

In that interval, Eq. (1) can be written as

$$y'' + p(x)y' + q(x)y = 0 \quad \dots (2)$$

where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$

are continuous functions. Therefore by Thm (3.2.1)

there exists a Unique solution that satisfies

Eq. (1) together with the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0.$$

To find such a solution in terms of power series,

we assume that the solution has the form:

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

which is defined on an Interval of Convergence

$|x-x_0| < \rho$, where $x_0 =$ Ordinary point.
 $\rho \rightarrow$ radius of Convergence

Example: Find Ordinary and Singular points of

$$1) (x^2 - x)y'' + xy' - 2x^2y = 0.$$

Sol: $P(x) = x^2 - x = x(x-1) = 0 \Rightarrow x = 0, x = 1$

are Singular points. So all other points real

or Complex are Ordinary points.

$$2) (x^2 + 4)y'' + xy = 0$$

Sol: $P(x) = x^2 + 4 = 0 \Rightarrow x = \pm 2i$ are Singular

points. All other points real or Complex are Ordinary.

Example: Find a series solution of

$$y'' + y = 0, \quad -\infty < x < \infty.$$

Sol: $P(x) = 1 \neq 0$, then every point is an ordinary

point, so we choose $x_0 = 0$ as a

simplest choice.

So, Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

$$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute y , y' & y'' in the D.E, we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+1)(n+2) a_{n+2} + a_n] x^n = 0$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + a_n = 0, \quad \forall n = 0, 1, 2, \dots$$

$$\Rightarrow a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad \forall n = 0, 1, 2, \dots$$

$$\text{So, if } n=0 \Rightarrow a_2 = \frac{-a_0}{(2)(1)} = \frac{-1}{2!} a_0$$

$$\text{if } n=1 \Rightarrow a_3 = \frac{-a_1}{(3)(2)} = \frac{-a_1}{3!}$$

$$\text{if } n=2 \Rightarrow a_4 = \frac{-a_2}{(4)(3)} = \frac{-(-a_0)}{(4)(3)(2)(1)} = \frac{+a_0}{4!}$$

$$\text{if } n=3 \Rightarrow a_5 = \frac{-a_3}{(5)(4)} = \frac{-(-a_1)}{(5)(4)(3!)} = \frac{a_1}{5!}$$

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 + a_1 x + \left(\frac{-a_0}{2!}\right)x^2 + \left(\frac{-a_1}{3!}\right)x^3 + \left(\frac{a_0}{4!}\right)x^4 + \left(\frac{a_1}{5!}\right)x^5 + \dots$$

$$= a_0 \left(1 - \frac{1}{2!}x^2 + \frac{x^4}{4!} + \dots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$$

$$= \underline{a_0 \cos x + a_1 \sin x}$$

Example: Find two linearly independent power series

solutions y_1 and y_2 of $y'' - xy = 0$, $-\infty < x < \infty$.

about the ordinary point $x_0 = 0$.

Sol: Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

shift Index

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow (2)(1)a_2 x^0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-1}] x^n = 0$$

recurrence relation ↓

$$\Rightarrow 2a_2 = 0 \quad \text{and} \quad (n+2)(n+1)a_{n+2} - a_{n-1} = 0$$

for $n = 1, 2, \dots$

so we have $a_2 = 0$, $a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$, $n = 1, 2, \dots$

If $n = 1 \Rightarrow a_3 = \frac{a_0}{(3)(2)} = \frac{a_0}{6}$

If $n = 2 \Rightarrow a_4 = \frac{a_1}{(4)(3)} = \frac{a_1}{12}$

If $n = 3 \Rightarrow a_5 = \frac{a_2}{(5)(4)} = 0$

If $n = 4 \Rightarrow a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{(6)(5)(6)} = \frac{a_0}{180}$

If $n = 5 \Rightarrow a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)} = \frac{a_1}{504}$

$$\begin{aligned}
 \therefore y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots \\
 &= a_0 + a_1 x + 0 + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 + 0 + \frac{a_0}{180} x^6 \\
 &\quad + \frac{a_1}{504} x^7 + \dots \\
 &= a_0 \left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \dots \right) + a_1 \left(x + \frac{x^4}{12} + \frac{x^7}{504} + \dots \right) \\
 &= a_0 \underbrace{\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \dots \right)}_{y_1} + a_1 \underbrace{\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \dots \right)}_{y_2}
 \end{aligned}$$

Now, to check that y_1 and y_2 are Linearly Independent

We compute $W(y_1, y_2)(0)$

$$\Rightarrow W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

$\therefore \{y_1, y_2\}$ forms a Fundamental Set of Solution.

Example: Solve $y'' - xy = 0$, $x_0 = 1$

Sol: Let $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$(x-1)+1$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=0}^{\infty} a_n (x-1)^{n+1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Shift Index

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\Rightarrow (2)(1) a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-1} - a_n] (x-1)^n = 0$$

$$\Rightarrow a_2 = \frac{a_0}{2} \quad \text{and} \quad a_{n+2} = \frac{a_n + a_{n-1}}{(n+2)(n+1)}, \quad n=1, 2, \dots$$

$$\text{If } n=1, \quad a_3 = \frac{a_1 + a_0}{(3)(2)} = \frac{a_0}{6} + \frac{a_1}{6}$$

$$\text{If } n=2, \quad a_4 = \frac{a_2 + a_1}{(4)(3)} = \frac{a_1}{12} + \frac{a_0}{(4)(3)(2)}$$

$$\text{If } n=3, \quad a_5 = \frac{a_3 + a_2}{(5)(4)} = \frac{a_3}{(5)(4)} + \frac{a_2}{(5)(4)}$$

$$\Rightarrow a_5 = \frac{a_0}{(6)(5)(4)} + \frac{a_1}{(6)(5)(4)} + \frac{a_0}{(5)(4)(2)}$$

⋮

$$y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \dots$$

$$= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \left(\frac{a_0}{6} + \frac{a_1}{6}\right)(x-1)^3$$

$$+ \left(\frac{a_1}{12} + \frac{a_0}{24}\right)(x-1)^4 + \dots$$

$$= a_0 \underbrace{\left(1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \dots\right)}_{y_1}$$

$$+ a_1 \underbrace{\left((x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \dots\right)}_{y_2}$$

$$= a_0 y_1 + a_1 y_2$$

$$W(y_1, y_2)(1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

∴ y_1 and y_2 are Linearly Independent.

So $\{y_1, y_2\}$ forms a F.S.S.

Q9) page 263. Find a power series solution for

$$(1+x^2)y'' - 4xy' + 6y = 0, \text{ about } x_0 = 0.$$

Sol: Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow (1+x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 4n a_n x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

Shift Index

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

$$\Rightarrow (2)(1)a_2 + (3)(2)a_3 x - 4a_1 x + 6a_0 + 6a_1 x$$

$$+ \sum_{n=2}^{\infty} \left[\underbrace{(n+2)(n+1)a_{n+2}}_{n+2} + \underbrace{n(n-1)a_n}_{n} - \underbrace{4na_n}_{n} + \underbrace{6a_n}_{n} \right] x^n = 0$$

$(n^2 - 5n + 6)$

$$\Rightarrow 2a_2 + 6a_0 + (6a_3 + 2a_1)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n-2)(n-3)a_n \right] x^n = 0$$

$$\Rightarrow 2a_2 + 6a_0 = 0, \quad 6a_3 + 2a_1 = 0$$

and $(n+2)(n+1)a_{n+2} + (n-2)(n-3)a_n = 0, \quad n=2, \dots$

\downarrow
recurrence relation.

$$\Rightarrow a_2 = -3a_0, \quad a_3 = -\frac{1}{3}a_1$$

$$\& a_{n+2} = \frac{-(n-2)(n-3)}{(n+2)(n+1)} a_n, \quad n=2, 3, \dots$$

If $n=2, a_4=0$. If $n=3, a_5=0$

If $n=4, a_6 = -\frac{1}{15}(0)=0$. If $n=5, a_7=0 \dots$

$$\therefore y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3}a_1 x^3 + 0 + 0 + \dots$$

$$= a_0 \left(1 - 3x^2 \right) + a_1 \left(x - \frac{x^3}{3} \right)$$

$$= a_0 y_1 + a_1 y_2$$

$$W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$