

Chapter 4. Higher Order Linear Equations.

4.1 General Theory of the n th order Linear Equations

Def: An n th order linear differential equation is an equation of the form.

$$L[y] := y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_1(t)y' + a_0y = g(t). \quad \text{--- (1)}$$

with corresponding homogeneous D.E.

$$L[y] = y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_1(t)y' + a_0y = 0 \quad \text{--- (2)}$$

Eq. (1) needs n th Initial Conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \dots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)} \quad \text{--- (3)}$$

Theorem (4.1.1). If the functions $a_{n-1}(t), \dots, a_1(t), a_0(t)$

and $g(t)$ are all continuous on the open interval

$I = (\alpha, \beta)$ such that $t_0 \in I$, then there

exists exactly one solution $y = \phi(t)$ of the D.E. (1)

that also satisfies the Initial Conditions (3).

Example: Determine the largest interval in which

the solution of the following IVP is certain to exist

$$(x-1)y^{(4)} + (x+1)y'' + (\tan x)y = 0$$

with $y(0) = 1$, $y'(0) = y''(0) = y'''(0) = 0$

Sol: $y^{(4)} + \frac{(x+1)}{(x-1)}y'' + \frac{\tan x}{(x-1)}y = 0$

\Rightarrow Interval: $\mathbb{R} \setminus \{1, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots\}$

\Rightarrow The largest Interval containing $t=1$ is $(-\frac{\pi}{2}, 1)$

The General solution for the homogeneous eq. (2)

is given by: $y_h = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$

where y_1, \dots, y_n are solutions of Eq. (2), and

c_1, c_2, \dots, c_n are arbitrary constants.

To find c_1, \dots, c_n , we use the Initial

Conditions in eq. (3).

Theorem (4.1.2): If the functions $a_0(t), \dots, a_{n-1}(t)$

in eq. (2) are continuous on an open interval

$I = (\alpha, \beta)$, and if the functions y_1, \dots, y_n

are solutions of Eq. (2) with $W(y_1, \dots, y_n)(t_0) \neq 0$

for some $t_0 \in I$, then every solution of Eq. (2) can

be expressed as a linear combination of y_1, \dots, y_n .

Def: A set of solutions y_1, y_2, \dots, y_n of eq. (2)

whose Wronskian is non zero is referred to

as a Fundamental set of solutions (FFS).

Example: show that $\{1, t, t^3\}$ form a FSS

of $t y^{(3)} - y'' = 0$.

Sol: Let $y_1 = 1, y_2 = t, y_3 = t^3$

\Rightarrow For $y_1 = 1$, $y_1' = 0$, $y_1'' = 0$, $y_1'''(0) = 0$

$\Rightarrow t(0) - 0 = 0$ (✓) $\Rightarrow y_1$ is a solution.

Now, $y_2 = t$, $y_2' = 1$, $y_2'' = 0$, $y_2''' = 0$

$\Rightarrow t(0) - 0 = 0$ (✓) $\Rightarrow y_2$ is a solution

$y_3 = t^3$, $y_3' = 3t^2$, $y_3'' = 6t$, $y_3''' = 6$

$\Rightarrow t(6) - 6t = 0$ (✓) $\Rightarrow y_3$ is a solution

$$W(1, t, t^3) = \begin{vmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{vmatrix} = 6t$$

$W(1, t, t^3)(1) = 6(1) = 6 \neq 0$

$\Rightarrow \{1, t, t^3\}$ form a (F.S.S).

Linear Dependence and Independent.

Def: A functions f_1, f_2, \dots, f_n are linearly

independent if

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$

$\Rightarrow c_1 = c_2 = \dots = c_n = 0.$

Def: A function f_1, \dots, f_n are Linearly Dependent

if there exists c_1, \dots, c_n not all zero such that

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$$

Example: $f_1 = 1, f_2 = 2+t, f_3 = 3-t^2$ are

linearly independent, since if

$$c_1 (1) + c_2 (2+t) + c_3 (3-t^2) = 0$$

$$(c_1 + 2c_2 + 3c_3) + c_2 t - c_3 t^2 = 0$$

$$\Rightarrow c_1 + 2c_2 + 3c_3 = 0 \quad \& \quad c_2 = 0, \quad c_3 = 0$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \Rightarrow \text{L. Independent.}$$

Example: Determine whether the functions

$$f_1(t) = 1, f_2(t) = t+2, f_3(t) = 3-t^2, f_4(t) = t^2+4t$$

are Linearly Independent or dependent.

Sol: Let $c_1 (1) + c_2 (t+2) + c_3 (3-t^2) + c_4 (t^2+4t) = 0$

$$\Rightarrow (c_1 + 2c_2 + 3c_3) + (c_2 + 4c_4)t + (c_4 - c_3)t^2 = 0$$

$$\Rightarrow c_1 + 2c_2 + 3c_3 = 0$$

$$c_2 + 4c_4 = 0$$

$$-c_3 + c_4 = 0$$

These three equations with four unknowns, have many solutions.

Since, if we set $c_4 = t \Rightarrow c_3 = t \Rightarrow c_2 = -4t$

$$\Rightarrow c_1 = 5t, \quad t \in \mathbb{R}$$

Thus $\{f_1, f_2, f_3, f_4\}$ are linearly dependent.

4.2 Homogeneous equations with Constant Coefficients

Consider the n th order linear homogeneous D.E:

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

where a_0, a_1, \dots, a_n are constants and $a_n \neq 0$.

To solve eq.(1), we use the same our knowledge of 2nd order linear D.Es considered in Ch 3.

Example: Solve $2y''' - 4y'' - 2y' + 4y = 0$

Characteristic eq.: $2r^3 - 4r^2 - 2r + 4 = 0$

$$\Rightarrow 2r^2(r-2) - 2(r-2) = (r-2)(2r^2-2) = 0$$

$$\Rightarrow r = 2, 1, -1$$

$$\therefore y_h = c_1 e^{2t} + c_2 e^t + c_3 e^{-t}$$

Example: Solve $y^{(4)} + 2y'' + y = 0$

$$\Rightarrow r^4 + 2r^2 + 1 = 0 \Leftrightarrow (r^2 + 1)(r^2 + 1) = 0$$

$$\therefore r_{1,2} = \pm i \quad \text{and} \quad r_{3,4} = \pm i$$

$$\therefore y(t) = (C_1 e^{0t} \cos t + C_2 e^{0t} \sin t) + (C_3 e^0 \cos t + C_4 \sin t)$$

$$\Rightarrow y(t) = C_1 \cos t + C_2 \sin t + C_3 t \cos t + C_4 t \sin t$$

Example: Solve $y^{(4)} - y = 0 \Rightarrow r^4 - 1 = 0$

$$\Leftrightarrow (r^2 + 1)(r^2 - 1) = 0 \Leftrightarrow r_{1,2} = \pm i, r_{3,4} = \pm 1$$

$$y(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$$

Example: Solve $y^{(4)} + y''' - 7y'' - y' + 6y = 0$

Sol: Char. Eq. : $r^4 + r^3 - 7r^2 - r + 6 = 0$

$$= (r-1)(r^3 + 2r^2 - 5r - 6)$$

$$= (r-1)(r+1)(r^2 + r - 6)$$

$$= (r-1)(r+1)(r-2)(r+3) = 0 \Rightarrow$$

$$r = 1, -1, 2, -3$$

$$\therefore y(t) = C_1 e^t + C_2 e^{-t} + C_3 e^{2t} + C_4 e^{-3t}$$

$$\begin{array}{r} r^3 + 2r^2 - 5r - 6 \\ (r-1) \overline{) r^4 + r^3 - 7r^2 - r + 6} \\ \underline{-r^4 + r^3} \\ 2r^3 - 7r^2 - r + 6 \\ \underline{-2r^3 + 2r^2} \\ -5r^2 - r + 6 \\ \underline{+5r^2 + 5r} \\ -6r + 6 \\ \underline{-6r + 6} \\ 0 \end{array}$$

(160)

4.3. NonHomogeneous equations with Constant Coefficients.

(Method of undetermined Coefficients).

Consider the n th order linear nonhomogeneous eq. with constant coefficients:

$$L[y] = a_n y^{(n)} + \dots + a_1 y' + a_0 y = g(t) \quad \dots (1)$$

We will use the method of undetermined coefficients to find y_p if g is constant, sin, cos, exp, polynomial or finite sums and products of these functions. as we did in Section 3.5.

Example: solve $y''' - 3y'' + 3y' - y = 4e^t$

char. eq: $r^3 - 3r^2 + 3r - 1 = 0$

$$\Rightarrow (r^3 - 1) - 3r(r - 1) = 0$$

$$\Rightarrow (r - 1)(r^2 + r + 1) - 3r(r - 1) = 0$$

$$\Rightarrow (r - 1)(r^2 + r + 1 - 3r) = 0$$

$$\Rightarrow (r - 1)(r^2 - 2r + 1) = 0$$

$$\Rightarrow (r - 1)(r - 1)(r - 1) = 0.$$

$$\therefore y_h = c_1 e^t + c_2 t e^t + c_3 t^2 e^t.$$

$$\text{Let } y_p = A e^t \cdot \boxed{t^3} = A t^3 e^t$$

To find y_p , find y_p'' , y_p'' , y_p' and substitute in the D.E.

Example: $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$

$$y_h : (r^4 + 2r + 1) = 0 \Leftrightarrow (r^2 + 1)^2 = 0$$

$$\therefore r_{1,2} = \pm i, \quad r_{3,4} = \pm i$$

$$\therefore y_h = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

$$\text{Let } y_p = (A \sin t + B \cos t) \cdot \boxed{t^2}$$

$$\Rightarrow y_p = A t^2 \sin t + B t^2 \cos t.$$

Example: $y''' - 4y' = t + 3 \cos t + e^{-2t}$.

y_h : $r^3 - 4r = r(r^2 - 4) = r(r-2)(r+2) = 0$

$\therefore r = 0, 2, -2 \Rightarrow y_h = C_1 + C_2 e^{2t} + C_3 e^{-2t}$.

To find y_p we have 3 sub-differential equations,

1) $y''' - 4y' = t \Rightarrow y_{p1} = (At + B) \cdot \boxed{t}$

2) $y''' - 4y' = 3 \cos t \Rightarrow y_{p2} = (C \sin t + D \cos t) \cdot \boxed{t^0}$

3) $y''' - 4y' = e^{-2t} \Rightarrow y_{p3} = E e^{-2t} \cdot \boxed{t}$

$\therefore y_p = y_{p1} + y_{p2} + y_{p3} = At^2 + Bt + C \sin t + D \cos t + E t e^{-2t}$

Example: $y^{(5)} + 4y''' = \cos^2 t - \sin^2 t = \cos(2t)$

y_h : $r^5 + 4r^3 = r^3(r^2 + 4) = 0 \Rightarrow r = 0, 0, 0, \pm 2i$

$\Rightarrow y_h = C_1 + C_2 t + C_3 t^2 + C_4 \cos 2t + C_5 \sin 2t$.

Let $y_p = (A \cos(2t) + B \sin 2t) \cdot \boxed{t}$

Example: $y^{(4)} + y = t$

$$y_h: r^4 + 1 = 0 \Rightarrow r^4 + 2r^2 + 1 = 2r^2$$

$$\Rightarrow (r^2 + 1)^2 = 2r^2$$

$$\Rightarrow (r^2 + 1)^2 - (\sqrt{2}r)^2 = 0$$

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$$\therefore (r^2 + 1 - \sqrt{2}r)(r^2 + 1 + \sqrt{2}r) = 0.$$

$$\therefore \underbrace{(r^2 - \sqrt{2}r + 1)}_{\swarrow} \underbrace{(r^2 + \sqrt{2}r + 1)}_{\searrow} = 0$$

$$r_{1,2} = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} \quad , \quad r_{3,4} = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2}$$

$$r_{1,2} = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i \quad , \quad r_{3,4} = -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$$

$$\therefore y_h = C_1 e^{\frac{1}{\sqrt{2}}t} \cos\left(\frac{1}{\sqrt{2}}t\right) + C_2 e^{\frac{1}{\sqrt{2}}t} \sin\left(\frac{1}{\sqrt{2}}t\right) \\ + C_3 e^{-\frac{1}{\sqrt{2}}t} \cos\left(\frac{1}{\sqrt{2}}t\right) + C_4 e^{-\frac{1}{\sqrt{2}}t} \sin\left(\frac{1}{\sqrt{2}}t\right)$$

Let $y_p = At + B$.

$$y_p' = A \quad , \quad y_p'' = 0 = y_p''' = y_p^{(4)} \dots$$

Example: $y^{(4)} + y''' = 1 - t^2 e^{-t}$

$y_h: r^4 + r^3 = r^3(r+1) = 0 \Rightarrow r = 0, 0, 0, -1$

$\therefore y_h = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t}$

We have 2. subdifferential equations.

1) $y^{(4)} + y''' = 1 \Rightarrow y_{p_1} = A \cdot t^3$

2) $y^{(4)} + y''' = -t^2 e^{-t} \Rightarrow y_{p_2} = (Bt^2 + Ct + D)e^{-t} \cdot t$

$\therefore y_p = y_{p_1} + y_{p_2} = At^3 + (Bt^3 + Ct^2 + Dt)e^{-t}$

Example: $y^{(4)} + 2y^{(3)} + 2y'' = t \sin t + 3te^{-t} + 4e^{-t} \sin t$

$y_h = c_1 + c_2 t + c_3 e^{-t} \cos t + c_4 e^{-t} \sin t$ (check!)

We have 3 subdifferential equations

1. $y^{(4)} + 2y^{(3)} + 2y'' = t \sin t \Rightarrow y_{p_1} = (At+B) \cos t + (Ct+D) \sin t$

2. $y^{(4)} + 2y^{(3)} + 2y'' = 3te^{-t} \Rightarrow y_{p_2} = (Et+F)e^{-t}$

3. $y^{(4)} + 2y^{(3)} + 2y'' = 4e^{-t} \sin t \Rightarrow y_{p_3} = (Ge^{-t} \sin t + He^{-t} \cos t) \cdot t$

$\therefore y_p = y_{p_1} + y_{p_2} + y_{p_3} \dots$

End of Chapter 4.