Chapter 2 : Groups.

Def : A binary operation : on a set G is a function that assigns to each ordered pairs of elements in G an element in G . (closed Abinary operation)

exp : on $\mathbb{Z}$. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$(m,n) \to m+n$

$m,n \in \mathbb{Z}$

$m+n,\ m.n \in \mathbb{Z}$

$g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$(m,n) \to m.n$

} → are binary operations.

$h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$(m,n) \to \frac{m}{n}$

$m,n \in \mathbb{Z}$ but $\frac{m}{n} \notin \mathbb{Z}$

} → is not a binary operations.

Def : Let G be a nonempty set, Let $*$ be abinary operation on G Then $(G,*)$ is called a group iff : ① closure.

1. **Associative :** $\forall a,b,c \in G \to (a*b)*c = a*(b*c)$.

2. **Identity :** $\exists e \in G,\ e*a = a \quad \forall a \in G$. (e called an Identity).

3. **Inverse :** $\forall a \in G, \exists b \in G,\ a*b = b*a = e$. (b called an Inverse for a).

$* : +, \cdot, -, \wedge$

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exp:

1. $(\mathbb{Z}, +)$ is a group. a. Associative : yes, $(a+b)+c = a+(b+c)$

b. Identity : 0 is the identity, $0+a = a$

c. Inverse : $-a$ is the inverse of a, $-a+a = 0$

2. $(\mathbb{R}^*, \cdot)$ is a group. a. Associative : $(a.b).c = a.(b.c)$

b. Identity : 1 is the identity.

c. Inverse : $a^{-1} = \frac{1}{a}$ is the inverse.

3. $(\mathbb{Q}^*, \cdot)$ is a group.

4. $(M_{2\times 2}, +)$ is a group.

5. $(GL(2,\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) / \det A \neq 0\}, \cdot)$ is a group.

a. Associative ✓

b. Identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c. Inverse : $A^{-1} = \frac{1}{\text{determenet}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

6. $(\{1, -1, i, -i\}, \cdot)$ is a group. $i = \sqrt{-1}$ $i^2 = -1$

$\odot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

الي بيخل على العين وخليه زي ما هو بيكون هو الـ Identity

Identity = 1

Inverse : 1 is 1

-1 is -1

i is $-i$

$-i$ is i

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exp: $(\mathbb{Z}_4, \oplus_4) = \{[0], [1], [2], [3]\}$

$\oplus_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Associtive.

Identity = 0

Inverse ⇒ 0 is 0
1 is 3
2 is 2
3 is 1

its a group.

بشوف الصف والعمود ما بتقاطعوا عند Identity ويكون هو Inverse

Note: $\mathbb{Z}$, R = relation.

$a \equiv b$ iff $4 / a-b$.

ref.

sym.

tran.

$[0] = \{\ldots -8, -4, 0, 4, 8, 12, \ldots\}$

$[1] = \{\ldots -1, -3, 1, 5, 9, 13 \ldots\}$

$[2] = \{\ldots -6, -2, 2, 6, 10, 14 \ldots\}$

$[3] = \{\ldots -5, -1, 3, 7, 11, 15 \ldots\}$

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exp: $(\mathbb{Z}_4, \otimes)$. its not a group.

exp: $(\mathbb{Z}_5^*, \otimes_5)$: $\{[1],[2],[3],[4]\}$

$\otimes$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

its a group.

Def : A group G is commutative (abelian) iff $a*b = b*a \quad \forall a,b,c \in G$.

$GL(2, \mathbb{R})$ is a nonabelian group.

Def : A group G is finite iff G has finite number of lements. otherwise G is infinite.

Thm : let $(G, *)$ be a group then e is unique.

pf: spse that $e, \bar{e}$ are two identites , Then

$$e = e * \bar{e} = \bar{e}$$

since $\bar{e}$ is the identity ↓ ↓ since e is the identity.

$$\Rightarrow \quad e = \bar{e}$$

so its one identity.

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Thm: If G is a group and $b \in G$ then b^{-1} is unique.

pf: spse $\bar{b}, \bar{\bar{b}}$ are two inverses for b. Then:

$$(\bar{b} * b) * \bar{\bar{b}} = \bar{b} * (b * \bar{\bar{b}})$$

$$e * \bar{\bar{b}} = \bar{b} * e$$

$$\bar{\bar{b}} = \bar{b} \quad \#$$

Thm: cancellation: let G be a group, let $a, b, c \in G$ Then

if $a * b = a * c$ then $b = c$.

if $b * a = c * a$ then $b = c$.

pf: spse, $a * b = a * c$

$$a^{-1}(a * b) = a^{-1}(a * c)$$

$$(a^{-1} * a) * b = (a^{-1} * a) * c$$

$$e * b = e * c$$

$$b = c \quad \#$$

exp. to groups:

1. $(\mathbb{Z}, +)$
2. $(\mathbb{Q}, +)$
3. $(\{\pm 1\}, \cdot)$
4. $(\mathbb{Q}^*, \cdot)$
5. $(GL(n, \mathbb{R}), \cdot)$, $GL(n, \mathbb{R}) = \{A \in M_{n \times n} \mid |A| \neq 0\}$

* $\mathbb{Z}_n :=$ the set of all equivalence classes mod n.

$= \{[0], [1], [2], \ldots [n-1]\}$.

Define $[x] + [y] := [x + y]$

Fact: $(\mathbb{Z}_n, +)$ forms a Group.

→ Question: Could we define $[x][y] = [xy]$ and make a group??

exp: $n = 6$

+	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	1	2	3	4	5
[1]	1	2	3	4	5	0
[2]	2	3	4	5	0	1
[3]	3	4	5	0	1	2
[4]	4	5	0	1	2	3
[5]	5	0	1	2	3	4

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examples :

① show that $GL((2,\mathbb{R}),.)$ is not commutative :

$$A=\begin{bmatrix}3 & 2\\ 4 & 2\end{bmatrix} \qquad B=\begin{bmatrix}5 & 3\\ 6 & 2\end{bmatrix}$$

Notice that $AB \neq BA$.

② Find $\begin{bmatrix}2 & 6\\ 3 & 5\end{bmatrix}^{-1}$ in $GL(2,\mathbb{R})$.

$$A^{-1}=\frac{1}{(2)(5)-(6)(3)}\begin{bmatrix}5 & -6\\ -3 & 2\end{bmatrix}$$

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exp. Which at the following is a group, Justify :

① $(Q^{+},.)$: Assosative ✓

identity $=1$

inverse of $\frac{m}{n}$ is $\frac{n}{m}$. so its a group.

② $(Q^{c+},.)$: Not Assosative, $\sqrt{2}\times\sqrt{2}=2\notin Q^{c}$

No Identity so not group.

③ $(M_{2\times2}(\mathbb{R}),.)$: No inverse For $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ so not group.

④ $(\mathbb{Z}_4, \otimes_4)$: 0, 2 not inverse so not group

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⑤ $\mathbb{R}^2 = (\{(a,b) / a,b \in \mathbb{R}\}, *)$

$(a+b) + (c,d) = (a+c, b+d)$

Assosiative ✓

identity $= (0,0)$.

inverse for $(a,b) = (-a,-b) \rightarrow (a,b)+(-a,-b) = (0,0) = e$

so its agroup.

⑥ $(SL(2,\mathbb{R}) = \{A \in M_{2\times 2} / |A| = 1\}, \cdot)$ its agroup.

⑦ $GL(2, \mathbb{Z}_7) \overset{\text{mod } 7}{=} \{A \mid \det A \neq 0\}$ Group.

exp $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = 3 - 8 = -5 \text{ mod } 7 = 2$

inverse: $\frac{1}{2}\begin{bmatrix} 1 & -2 \\ -4 & 3 \end{bmatrix} \overset{\text{mod } 7}{=} 4\begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix} \overset{\text{بدل}}{=} \begin{bmatrix} 4 & 20 \\ 12 & 12 \end{bmatrix}$

كان مرة mod 7 $= \begin{bmatrix} 4 & 6 \\ 5 & 5 \end{bmatrix}$

للتأكيد ضرب الحل بالجواب الاخير بالمصفوفة الوحدة وتطلع $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ch2 Done

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