2.3 Additional Topics and Applications

In this section, we learn a method for computing the inverse of a nonsingular matrix A using determinants and we learn a method for solving linear systems using determinants. Both methods depend on Lemma 2.2.1.

The Adjoint of a Matrix

Let A be an $n \times n$ matrix. We define a new matrix called the *adjoint* of A by

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

Thus, to form the adjoint, we must replace each term by its cofactor and then transpose the resulting matrix.

By Lemma 2.2.1,

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and it follows that

$$A(\operatorname{adj} A) = \det(A)I$$

If A is nonsingular, $\det(A)$ is a nonzero scalar, and we may write

$$A\left(\frac{1}{\det(A)} \operatorname{adj} A\right) = I$$

A^{-1}

Thus,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A \quad \text{when } \det(A) \neq 0$$

EXAMPLE 1 For a 2×2 matrix,

$$\operatorname{adj} A = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

If A is nonsingular, then

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$ ■

EXAMPLE 2 Let

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Compute $\operatorname{adj} A$ and A^{-1}.

Solution

$$\operatorname{adj} A = \begin{pmatrix} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{pmatrix}^T = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A = \frac{1}{5} \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

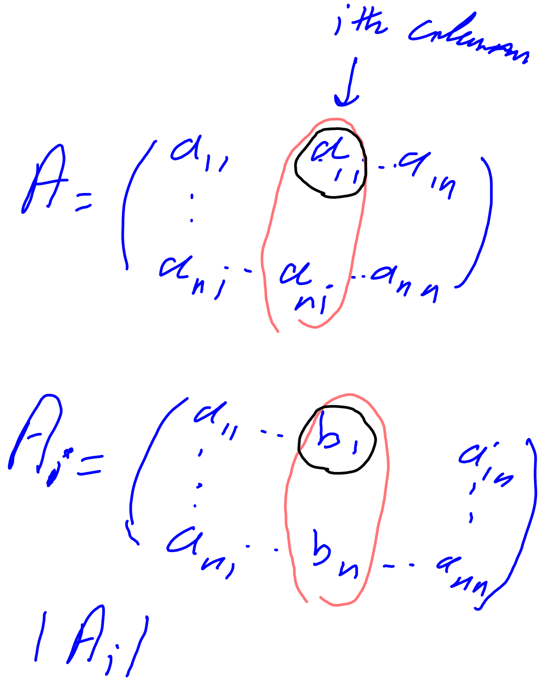
Theorem 2.3.1 Cramer's Rule

Let A be a nonsingular n × n matrix, and let $\mathbf{b} \in \mathbb{R}^n$. *Let* A_i *be the matrix obtained by replacing the ith column of A by* $\mathbf{b}$. *If* $\mathbf{x}$ *is the unique solution of* $A\mathbf{x} = \mathbf{b}$, *then*

$$x_i = \frac{\det(A_i)}{\det(A)} \quad \text{for} \quad i = 1, 2, \ldots, n$$

Proof Since

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}(\operatorname{adj} A)\mathbf{b}$$



it follows that

$$x_i = \frac{b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}}{\det(A)} = \frac{\det(A_i)}{\det(A)}$$

∎

EXAMPLE 3 Use Cramer's rule to solve

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix} \iff \left\{ \begin{aligned} x_1 + 2x_2 + x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 9 \end{aligned} \right.$$

$Ax = b$

Solution

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -4 \qquad \det(A_1) = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = -4$$

$$\det(A_2) = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = -4 \qquad \det(A_3) = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = -8$$

Therefore,

$$x_1 = \frac{-4}{-4} = 1, \qquad x_2 = \frac{-4}{-4} = 1, \qquad x_3 = \frac{-8}{-4} = 2$$

EXERCISES

8. Let A be a nonsingular $n \times n$ matrix with $n > 1$. Show that

$$\det(\operatorname{adj} A) = (\det(A))^{n-1}$$

10. Show that if A is nonsingular, then $\operatorname{adj} A$ is nonsingular and

$$(\operatorname{adj} A)^{-1} = \det(A^{-1})A = \operatorname{adj} A^{-1}$$

11. Show that if A is singular, then $\operatorname{adj} A$ is also singular.

12. Show that if $\det(A) = 1$, then

$$\operatorname{adj}(\operatorname{adj} A) = A$$