Ch 5 

## 5.1 Review of Power Series

- In this chapter we will learn how to find power series solution for some $2^{nd}$ order linear DE's

- The reason for that because some of these DE's could be with non constant coefficients.

**Exp** Solve the DE: $y'' + y = 0$

Ch. Eq. $r^2 + 1 = 0$

$r_{1,2} = \pm i$, $\lambda = 0$, $\mu = 1$

$y_1(x) = \cos x$ and $y_2(x) = \sin x$

gen. sol. $\Rightarrow$ $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$y(x) = c_1 \cos x + c_2 \sin x$$

$$y(x) = c_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}} + c_2 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

Power Series Solution

Fundemental Power Series Solutions about $x_0 = 0$

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**Remark** The Power Series Solution about $x_0$ for a given DE has the form:

center

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

Question: Why Power Series Solution ?

Answer: Exp Solve the DE : $y'' + xy = 0$

we can not use ch1 , nor ch2 (missing x and y), nor ch3 (since it is not constant coefficients , nor Euler DE, nor ch4 ...

since it is not

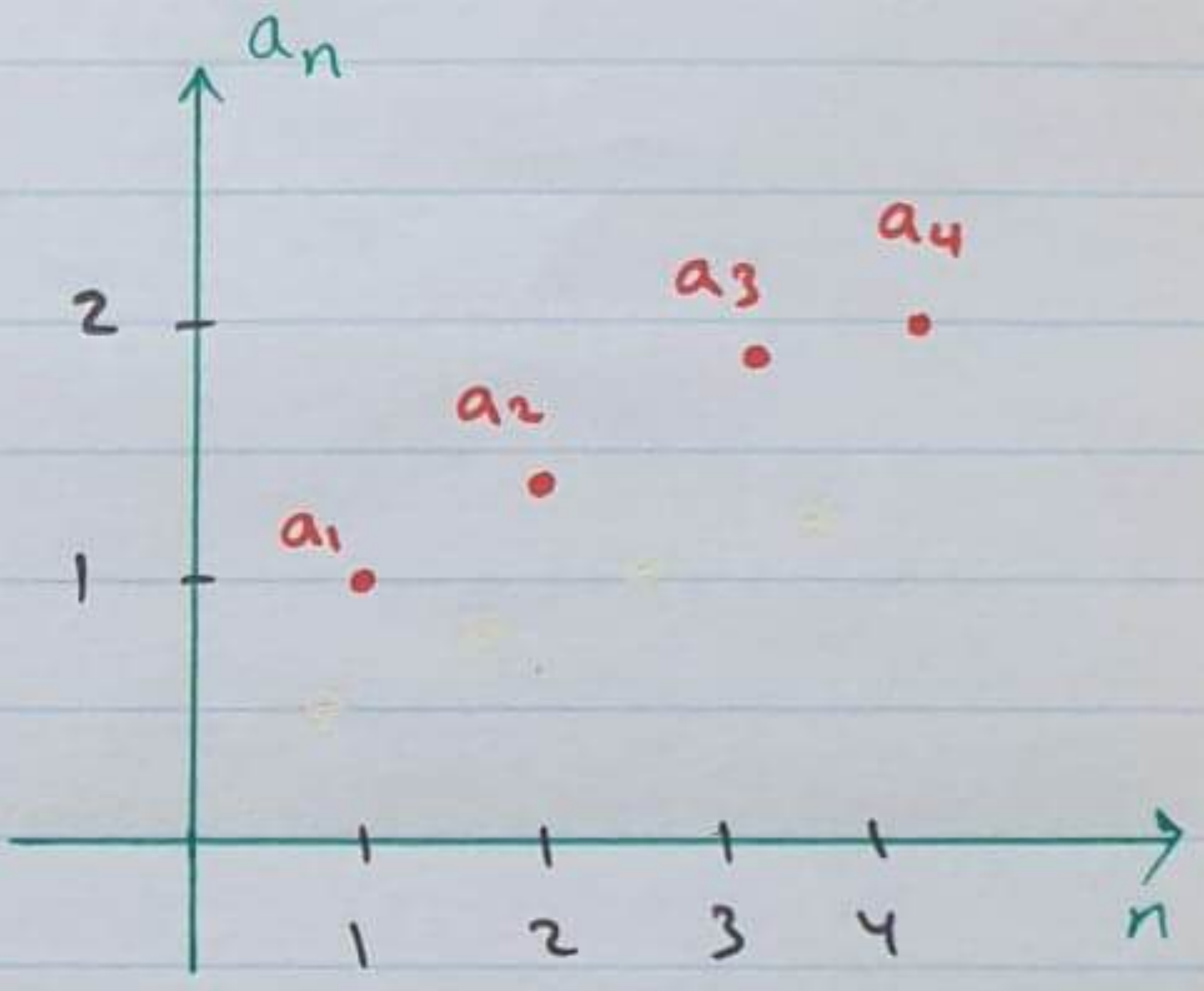
so we need ch5

## Review of Sequences:

Exp $a_n = \sqrt{n}$ , $n = 1, 2, 3, \ldots$

$a_1 = \sqrt{1} = 1$
$a_2 = \sqrt{2}$
$a_3 = \sqrt{3}$
$\vdots$

The sequence diverges since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt{n} = \infty$

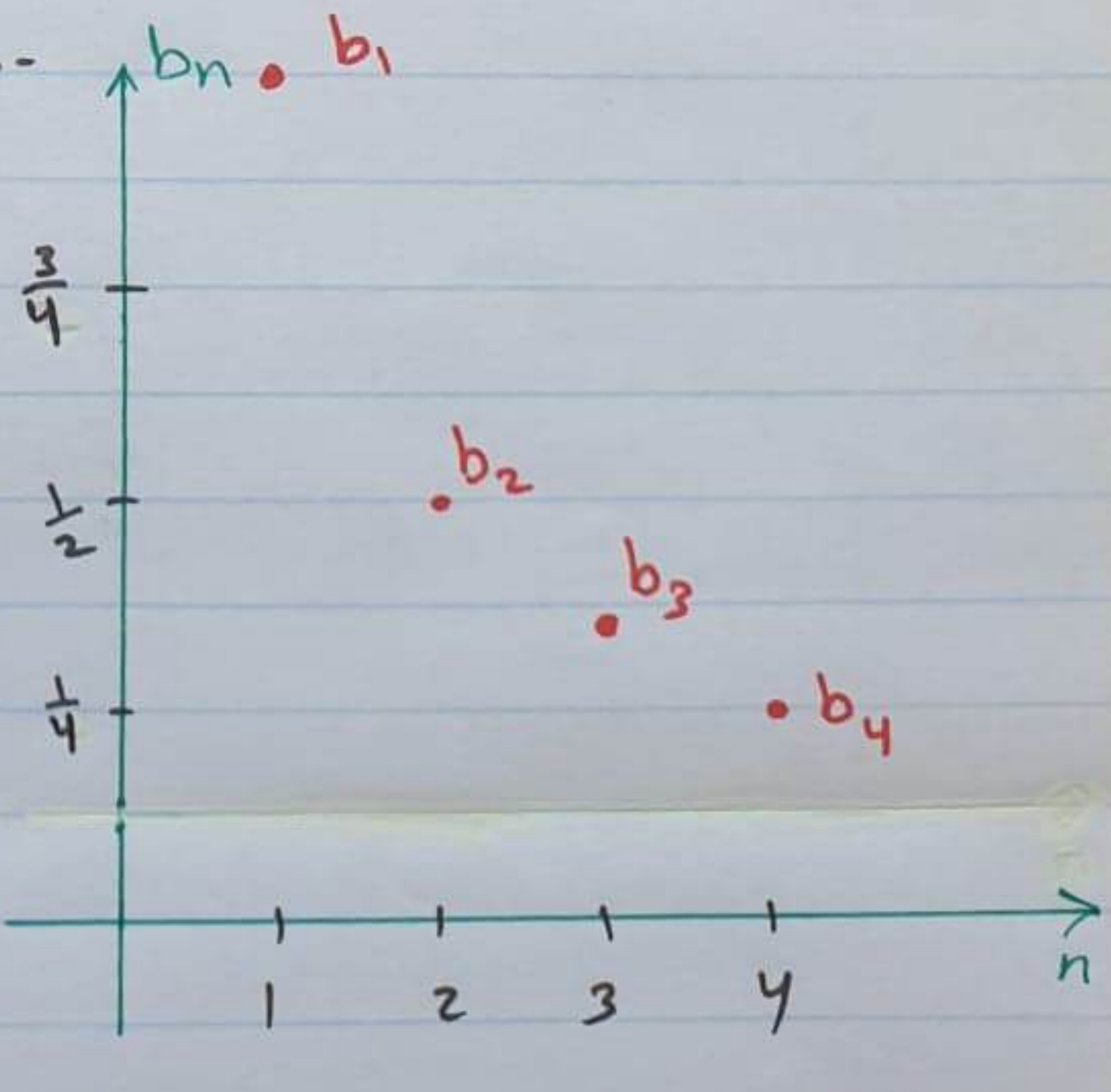



Exp $b_n = \frac{1}{n}$ , $n = 1, 2, 3, \ldots$

$b_1 = 1$
$b_2 = \frac{1}{2}$
$b_3 = \frac{1}{3}$
$\vdots$

The sequence converges to 0 since

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$



- Recall Taylor Series Expansion for an infinitly many differentiable function $f(x)$ about the point $x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

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- when $x_0 = 0$ Taylor Series is called Maclurine Series

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- $e^x$, $\sin x$, $\cos x$ are examples of analytic functions since they have Taylor Series Expansion everywhere " at any point $x_0$ "

- $f(x) = \frac{1}{x}$ is analytic everywhere except at $x = 0$

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- To solve DE's using the idea of finding power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{about } x_0 \quad \Rightarrow$$

we need to check the convergence of this power series solution $\Rightarrow$ so we may apply Ratio Test (RT) as follows:

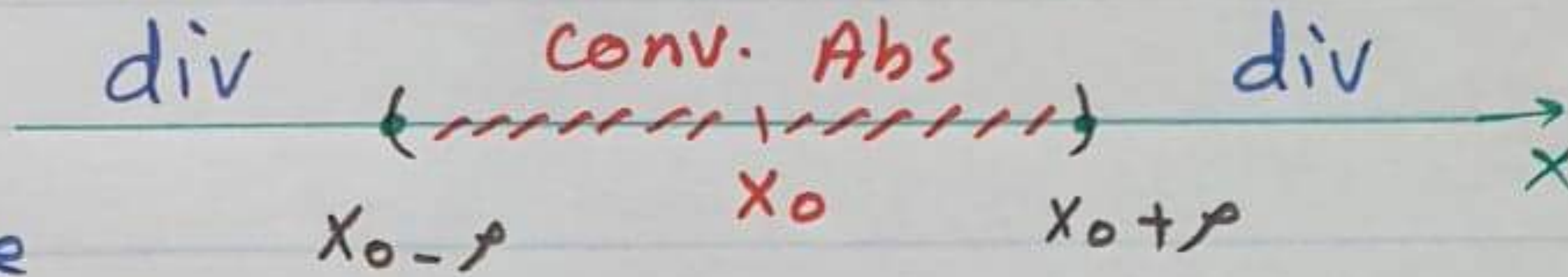
Assume $\lim\limits_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = L$ , where $b_n = a_n (x-x_0)^n$

1. If $L < 1$, then the power series converges

2. If $L > 1$, then the power series diverges

3. If $L = 1$, then the test fails

The power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ will converge absolutely for every $x$ belongs to the interval $|x - x_0| < \rho$




$\rho$ : Radius of Convergence
IC : Interval of Convergence

we check the endpoints for conditional Convergence.

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Exp Find $\rho$ and IC for the following power series:

[1] $\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n \Rightarrow x_0 = 2$

Apply RT $\Rightarrow$ $L = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2} (n+1)(x-2)^{n+1}}{(-1)^{n+1} n (x-2)^n} \right|$

$= |x-2| \lim_{n\to\infty} \left( \frac{n+1}{n} \right) = |x-2| (1)$

$= |x-2| < 1$

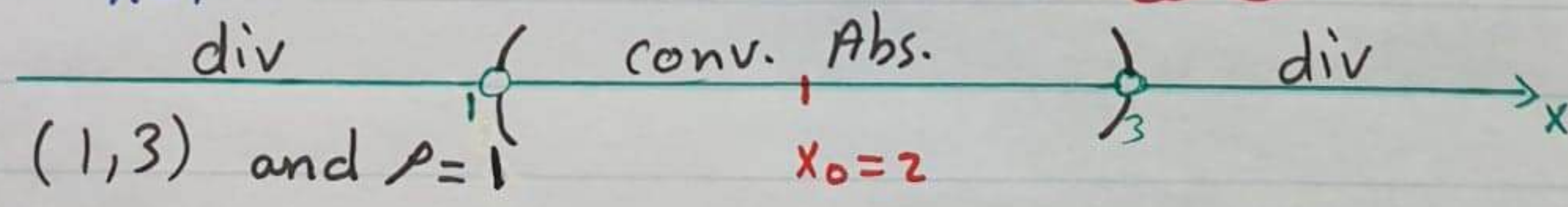
$-1 < x - 2 < 1$

$1 < x < 3$

The power series converges Absolutely on $(1, 3)$

- when $x = 1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} n (1-2)^n = \sum_{n=1}^{\infty} (-1)^{2n+1} n$
- when $x = 3 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} n (3-2)^n = \sum (-1)^{n+1} n \longrightarrow$

which diverges by $n^{th}$ term test as $\lim_{n\to\infty} n \neq 0$




Hence, IC $= (1,3)$ and $\rho = 1$

$\boxed{2}$ $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\,2^n} \Rightarrow x_0 = -1$

Apply RT $\Rightarrow$ $L = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n\,2^n}{(x+1)^n} \right|$

$$= \frac{|x+1|}{2} \lim_{n\to\infty} \left(\frac{n}{n+1}\right) = \frac{|x+1|}{2}(1) < 1$$

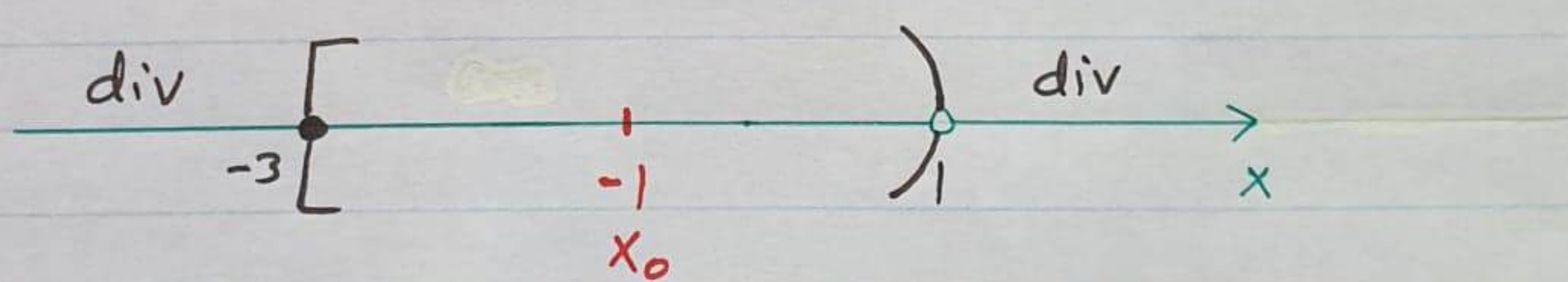
$$|x+1| < 2$$

$$-2 < x+1 < 2$$

$$-3 < x < 1$$

The power series converges Abs. on $(-3, 1)$

- when $x = -3 \Rightarrow \sum_{n=1}^{\infty} \frac{(-3+1)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ convergent Alternating Series
- when $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(1+1)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ divergent Harmonic series




Hence, $IC = [-3, 1)$ and $\rho = 2$

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The power series in this Exp Converges conditionally at $x = -3$ $\Rightarrow$ This means

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The power series converges at $x = -3$ but not Absolutely.

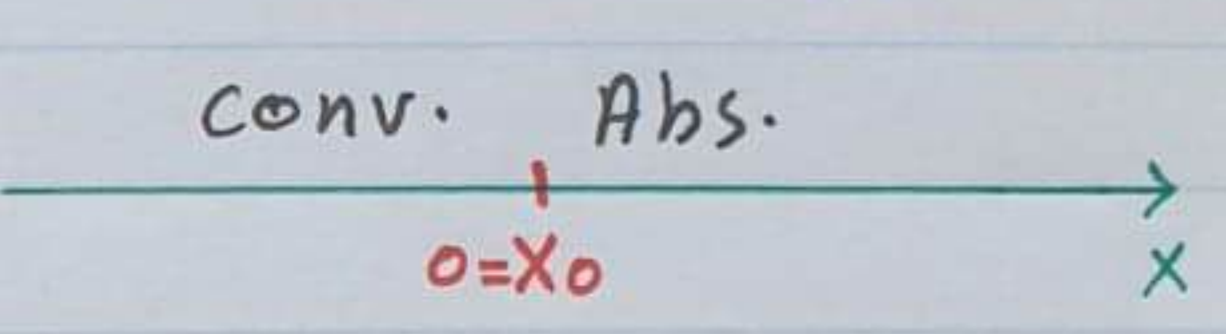
$\boxed{3}$ $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\Rightarrow x_0 = 0$

Apply RT $\Rightarrow L = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$

$= |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| (0) = 0 < 1$ ✓

Hence, this power series converges Abs. for every $x$

$IC = \mathbb{R} = (-\infty, \infty)$ with $\rho = \infty$




Note that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ "Maclurine Series of $e^x$"

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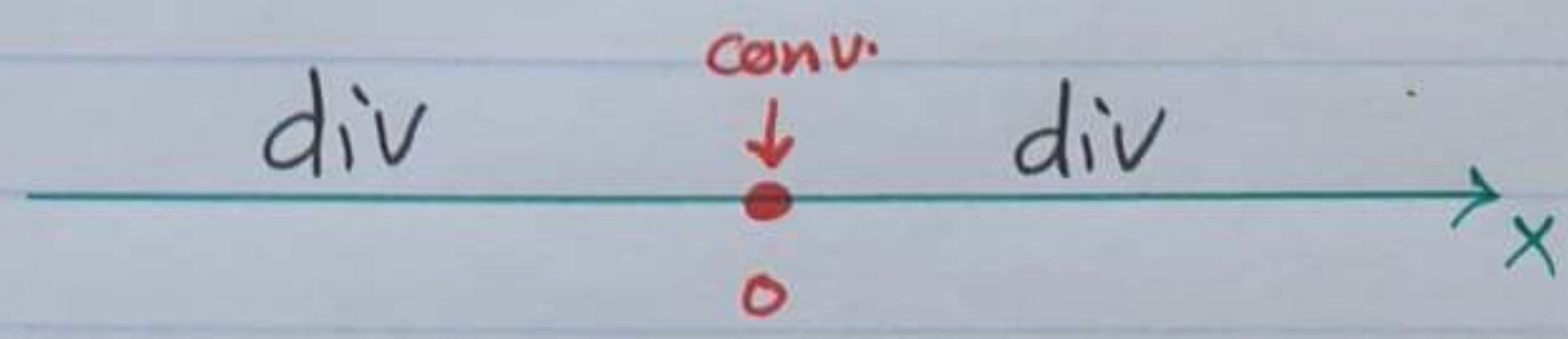
$\boxed{4}$ $\sum_{n=0}^{\infty} n!\, x^n$ $\Rightarrow x_0 = 0$

Apply RT $\Rightarrow \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right|$

$= |x| \lim_{n\to\infty} (n+1) = \infty > 1$ if $x \neq 0$ and so it diverges

If $x = 0 \Rightarrow \sum_{n=0}^{\infty} n!\, 0^n = 0 < 1$ and so it converges

Hence, $\sum_{n=0}^{\infty} n!\, x^n$ diverges for every $x \in \mathbb{R} \setminus \{0\}$




$\Rightarrow \rho = 0$ and the power series converges only at $x = 0$

Derivatives of the power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

$$y'(x) = \sum_{n=1}^{\infty} n\, a_n (x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + \cdots$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n (x-x_0)^{n-2} = 2a_2 + 3(2)a_3(x-x_0) + \cdots$$

Shifting Index :

It is not important which index we use in the upper and lower limits of the sum . That is

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{k=0}^{\infty} a_k (x-x_0)^k = \sum_{m=-1}^{\infty} a_{m+1} (x-x_0)^{m+1}$$

$$= \sum_{n=10}^{\infty} a_{n-10} (x-x_0)^{n-10}$$

Exp Rewrite the following power series involving the power of $(x-2)^n$

① $$\sum_{n=0}^{\infty} a_n (x-2)^{n+1} = \sum_{n=1}^{\infty} a_{n-1} (x-2)^n$$

② $$\sum_{n=0}^{\infty} n\, a_n (x-2)^{3+n} = \sum_{n=3}^{\infty} (n-3) a_{n-3} (x-2)^n$$

③ $$\sum_{k=5}^{\infty} k(k-1)(k-2)(x-2)^{k-3} = \sum_{n=2}^{\infty} (n+3)(n+2)(n+1)(x-2)^n$$

# $\boxed{5.2}$ Series Solution Near an Ordinary Point $x_0$ $\boxed{131}$
"Part I"

- Given the DE

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad \dots (*)$$

where $P, Q, R$ are polynomials

- Note that $(*)$ is $2^{nd}$ order linear homogeneous DE with variable coefficients

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**Def.** The DE $(*)$ has an Ordinary Point $x_0$ iff $P(x_0) \neq 0$

- The DE $(*)$ has Singular Point $z_0$ iff $P(z_0) = 0$

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**Exp** ① The DE $(x^2-4)y'' + (\sin x)y' - e^x y = 0$
has two singular points $\Rightarrow$

$$P(x) = x^2 - 4 = 0$$
$$(x-2)(x+2) = 0$$
$$x = 2 \text{ or } x = -2$$

All other points are ordinary "$\mathbb{R} \setminus \{-2, 2\}$"

② $(\ln x)y'' - xy' + y = 0$
has only on singular point $\Rightarrow$

$$P(x) = \ln x = 0$$
$$x = 1$$

All other points are ordinary "$\mathbb{R} \setminus \{1\}$"

③ $y'' - e^x y' + y = 0$
has no singular points $\Rightarrow$ All points are ordinary

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- Assume $x_0$ is an Ordinary Point (OP) for the DE (*):

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

- Hence, $P(x_0) \neq 0$.

- Let $p(x) = \dfrac{Q(x)}{P(x)}$ and $q(x) = \dfrac{R(x)}{P(x)}$

- Note that $p(x)$ and $q(x)$ are well-defined at the OP $x_0$. Moreover, $p(x)$ and $q(x)$ are analytic at $x_0$. That is, $p(x)$ and $q(x)$ have Taylor Series Expansion about the OP $x_0$:

$$p(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n$$

- Now divide the DE (*) by $P(x)$ $\Rightarrow$

(*') $$y'' + p(x)\,y' + q(x)\,y = 0 \quad , \; y(x_0) = y_0, \; y'(x_0) = y_0'$$

where $p(x)$ and $q(x)$ are cont. on an open interval $I$ about $x_0$

- By Th3.2.1 $\Rightarrow$ $\exists$ a unique solution $y(x)$ satisfies the IVP (*') on $I$.

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In this section we will find a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

about the OP $x_0$ for the DE (*).

• To find two independent power series solutions $y_1(x)$ and $y_2(x)$ $\Rightarrow$ we write the coefficients

$a_2, a_3, a_4, \dots$ interms of $a_0$ or $a_1$ so that

the power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \dots$$

$$= a_0 \, y_1(x) + a_1 \, y_2(x)$$

Then we check $w(y_1(x), y_2(x))(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}$

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<u>Exp</u> Find a series solution $\sum_{n=0} a_n x^n$ for the DE

$$y'' + y = 0 \quad , \quad x \in \mathbb{R}$$

• Comparing $\sum_{n=0}^{\infty} a_n x^n$ with $\sum_{n=0} a_n (x - x_0)^n$ $\Rightarrow$ $x_0 = 0$ is an OP

since $P(x) = 1$ and so all points are ordinary

• Our power series solution is then given by

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n \, a_n x^{n-1}$$

$$\Rightarrow y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) \, x^{n-2}$$

• Substitute $y''$ and $y$ into the DE $\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^{n} = 0$$

not          same power

① same power
② same index

shifting index

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^{n} + \sum_{n=0}^{\infty} a_n\, x^{n} = 0$$

same index

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} + a_n\right] x^{n} = 0$$

Comparing the coefficients of $x^n$ $\Rightarrow$

$$(n+2)(n+1)\, a_{n+2} + a_n = 0 \quad , \quad n = 0, 1, 2, \ldots$$

Recurrence Relation (RR)

$$\boxed{a_{n+2} = \frac{-a_n}{(n+1)(n+2)} \; , \; n = 0, 1, 2, \ldots}$$

We use RR to write $a_2, a_3, \ldots$ interms of $a_0$ and $a_1$ $\Rightarrow$

$$n=0 \Rightarrow a_2 = \frac{-a_0}{(1)(2)} = \frac{-a_0}{2!}$$

$$n=1 \Rightarrow a_3 = \frac{-a_1}{(2)(3)} = \frac{-a_1}{3!}$$

$$n=2 \Rightarrow a_4 = \frac{-a_2}{(3)(4)} = \frac{-\,\frac{-a_0}{2!}}{(3)(4)} = \frac{a_0}{4!}$$

$$n=3 \Rightarrow a_5 = \frac{-a_3}{(4)(5)} = -\frac{\frac{-a_1}{3!}}{(4)(5)} = \frac{a_1}{5!}$$

$$n=4 \Rightarrow a_6 = \frac{-a_4}{(5)(6)} = -\frac{\frac{a_0}{4!}}{(5)(6)} = -\frac{a_0}{6!}$$

The series solution is

$$y(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 - \frac{a_0}{6!}x^6 + \cdots$$

$$= a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] + a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]$$

$$= a_0 \; y_1(x) + a_1 \; y_2(x)$$

$$= a_0 \cos x + a_1 \sin x$$

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Note that the power series solutions $y_1(x)$ and $y_2(x)$ are L. Indep. since

$$W(y_1(x), y_2(x))(x_0) = W(y_1(x), y_2(x))(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Hence, they form fondemental series solutions

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Note also that this DE: $y'' + y = 0$ can be easily solved as follows: Ch. Eq. $r^2 + 1 = 0$

$r_{1,2} = \pm i$ $\qquad \lambda = 0$

$\mu = 1$

$$y_1(x) = \cos x \quad \text{and} \quad y_2(x) = \sin x$$

Hence, the gen. sol. is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$= c_1 \cos x + c_2 \sin x$

**Exp** Find two indep. power series solutions in the power of x for the DE $y'' - xy = 0$

- The series solution is $y(x) = \sum_{n=0}^{\infty} a_n x^n$ so $x_0 = 0$ is OP since $P(x) = 1$ and so all points are ordinary

- $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

- Substitute $y''$ and $y$ in the DE above $\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n \underset{\substack{\uparrow \\ \text{not}}}{x^{n-2}} - \sum_{n=0}^{\infty} a_n \underset{\substack{\uparrow \\ \text{not same power}}}{x^{n+1}} = 0$$

✓ ① same power

✓✓ ② same index

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

not same index

$$(2)(1) a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2 a_2 + \sum_{n=1}^{\infty} \left[ (n+2)(n+1) a_{n+2} - a_{n-1} \right] x^n = 0$$

$\boxed{a_2 = 0}$ and $\boxed{a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}, \quad n = 1, 2, 3, \ldots}$ (RR) Recurrence Relation we use RR to write $a_2, a_3, \dots$ interms of $a_0$ and $a_1 \Rightarrow$

$$n=1 \Rightarrow a_3 = \frac{a_0}{(2)(3)} = \frac{a_0}{3!}$$

$$n=2 \Rightarrow a_4 = \frac{a_1}{(3)(4)} = \frac{2a_1}{4!}$$

$$n=3 \Rightarrow a_5 = \frac{a_2}{(4)(5)} = 0$$

$$n=4 \Rightarrow a_6 = \frac{a_3}{(5)(6)} = \frac{\frac{a_0}{(2)(3)}}{(5)(6)} = \frac{a_0}{(2)(3)(5)(6)} = \frac{4a_0}{6!}$$

$$n=5 \Rightarrow a_7 = \frac{a_4}{(6)(7)} = \frac{a_1/(3)(4)}{(6)(7)} = \frac{a_1}{(3)(4)(6)(7)} = \frac{10a_1}{7!}$$

$$n=6 \Rightarrow a_8 = \frac{a_5}{(7)(8)} = 0$$

$\vdots$

The series solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cancel{a_2 x^2} + a_3 x^3 + a_4 x^4 + \cancel{a_5 x^5} + a_6 x^6 + \dots$$

$$= a_0 + a_1 x + 0 + \frac{a_0}{3!} x^3 + \frac{2a_1}{4!} x^4 + 0 + \frac{4a_0}{6!} x^6 + \frac{10a_1}{7!} x^7 + \dots$$

$$= a_0 \left[1 + \frac{x^3}{3!} + \frac{4x^6}{6!} + \dots\right] + a_1 \left[x + \frac{2x^4}{4!} + \frac{10x^7}{7!} + \dots\right]$$

$$= a_0 \; y_1(x) \; + \; a_1 \; y_2(x)$$

$y_1(x)$ and $y_2(x)$ are the two indep. power series solutions since

$$W(y_1(x), y_2(x))(x_0) = W(y_1(x), y_2(x))(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Hence they also form Fundemental series solutions.

**Exp** Find Fundemental series solutions for the DE: $y'' - xy = 0$ about $x_0 = 1$

- $P(x) = 1$ never zero so all points are ordinary $\Rightarrow x_0 = 1$ is OP
- The series solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

- substitute $y''$ and $y$ in the DE $\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \big((x-1)+1\big) \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n \underset{\uparrow \text{not}}{(x-1)^{n-2}} - \sum_{n=0}^{\infty} a_n \underset{\uparrow \text{not}}{(x-1)^{n+1}} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

① same power
② same index

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$(2)(1) a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - a_0 - \sum_{n=1}^{\infty} a_n (x-1)^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \Big[ (n+2)(n+1) a_{n+2} - a_{n-1} - a_n \Big] (x-1)^n = 0$$

Comparing coefficients $\Rightarrow$

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$$

$$\boxed{a_{n+2} = \frac{a_{n-1} + a_n}{(n+1)(n+2)} \quad , n = 1,2,3,\ldots} \text{—(RR)}$$

Recurrence Relation

We use RR to write the coefficients $a_3, a_4, a_5, \ldots$ interms of $a_0$ and $a_1 \Rightarrow$

$$n=1 \Rightarrow a_3 = \frac{a_0 + a_1}{(2)(3)} = \frac{a_0}{3!} + \frac{a_1}{3!}$$

$$n=2 \Rightarrow a_4 = \frac{a_1 + a_2}{(3)(4)} = \frac{2a_1}{4!} + \frac{a_0/2}{(3)(4)} = \frac{2a_1}{4!} + \frac{a_0}{4!}$$

$$n=3 \Rightarrow a_5 = \frac{a_2 + a_3}{(4)(5)} = \frac{a_0/2 + a_0/3! + a_1/3!}{(4)(5)} = \frac{4a_0}{5!} + \frac{a_1}{5!}$$

$\vdots$

The series solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots$$

$$= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \left(\frac{a_0}{3!} + \frac{a_1}{3!}\right)(x-1)^3 + \left(\frac{2a_1}{4!} + \frac{a_0}{4!}\right)(x-1)^4 + \cdots$$

$$= a_0\left[1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots\right] + a_1\left[(x-1) + \frac{(x-1)^3}{3!} + \frac{2(x-1)^4}{4!} + \cdots\right]$$

$$= a_0 \quad y_1(x) \quad + \quad a_1 \quad y_2(x)$$

$$W(y_1(x), y_2(x))(1) = \begin{vmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$y_1$ and $y_2$ are L. Indep. Thus, they form fundemental set of solutions.

Exp Find power series solution for the DE

$$(1-x^2)y'' - 2xy' + 6y = 0 \quad \text{about } x_0 = 0$$

- $P(x) = 1 - x^2 = 0 \iff x = \pm 1$ singular Points

Hence, $x_0 = 0$ is OP

- The series solution is $y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$
- substitute $y'', y', y$ in the DE

$$\Rightarrow y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

✓ same power

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{\substack{n=2 \\ n=0}}^{\infty} n(n-1)a_n x^n - \sum_{\substack{n=1 \\ n=0}}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

✓✓ same index

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n a_n + 6a_n \right] x^n = 0$$

$$-a_n(n^2 - n + 2n - 6)$$

$$-a_n(n^2 + n - 6)$$

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1)a_{n+2} - a_n (n-2)(n+3) \right] x^n = 0$$



$\Downarrow$

$$\boxed{a_{n+2} = \frac{(n-2)(n+3)\,a_n}{(n+1)(n+2)}, \quad n = 0, 1, 2, \dots}$$ → Recurrence Relation (RR)

We use RR to write $a_2, a_3, a_4, \dots$ interms of $a_0$ and $a_1$ as follow $\Rightarrow$

$$n=0 \Rightarrow a_2 = \frac{(-\cancel{2})(3)\,a_0}{(1)\cancel{(2)}} = -3a_0$$

$$n=1 \Rightarrow a_3 = \frac{(-1)(\cancelto{2}{4})\,a_1}{\cancel{(2)}(3)} = -\frac{2}{3}a_1$$

$$n=2 \Rightarrow a_4 = 0$$

$$n=3 \Rightarrow a_5 = \frac{(1)(\cancelto{3}{6})\,a_3}{(\cancelto{2}{4})(5)} = \frac{3\left(-\frac{2}{3}a_1\right)}{(2)(5)} = -\frac{1}{5}a_1$$

$$n=4 \Rightarrow a_6 = 0$$

$$n=5 \Rightarrow a_7 = \frac{(\cancel{3})(\cancelto{4}{8})\,a_5}{(\cancelto{3}{6})(7)} = \frac{4\left(-\frac{1}{5}a_1\right)}{7} = -\frac{4}{35}a_1$$

$$n=6 \Rightarrow a_8 = 0$$

$$\vdots$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cancelto{0}{a_4 x^4} + a_5 x^5 + \cancelto{0}{a_6 x^6} + \dots$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_5 x^5 + a_7 x^7 + a_9 x^9 + \dots$$

$$= a_0 + a_1 x - 3a_0 x^2 - \frac{2}{3}a_1 x^3 - \frac{1}{5}a_1 x^5 - \frac{4}{35}a_1 x^7 + \dots$$

$$= a_0\left[1 - 3x^2\right] + a_1\left[x - \frac{2}{3}x^3 - \frac{1}{5}x^3 - \frac{4}{35}x^7 + \dots\right]$$

$$= a_0\, y_1(x) + a_1\, y_2(x)$$

# 5.3 Series Solution about Ordinary Point II

Recall from 5.2 that the series solution about the OP $x_0$ has the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for a given DE of the form

$$\boxed{P(x) y'' + Q(x) y' + R(x) y = 0} \quad *$$

where $P(x)$, $Q(x)$, $R(x)$ are polynomials.

Question what happen if $P(x), Q(x), R(x)$ not all poly. ?

Answer: It will be hard to find series solution as in 5.2 procedure.

Exp Find series solution of power $x$ for the DE

$$(x+1) y'' - \ln(e + x^2) y' - 2y = 0$$

Note that $Q(x) = -\ln(e + x^2)$ is not poly.

series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ since $x_0 = 0$ is OP.
since $P(0) = 1 \neq 0$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

substitute $y, y', y''$ in the DE $\Rightarrow$

$$(x+1) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \underbrace{- \ln(e + x^2) \sum n a_n x^{n-1}}_{\text{Problem}} - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

We need new method to solve Exp'

---

Question Given the IVP:

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad , \; y(x_0) = y_0 \; , \; y'(x_0) = y'_0$$

where $x_0$ is an OP and $P(x), Q(x), R(x)$ are functions having all derivative at $x_0$.

Show that if $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots$

is a power series solution to this IVP about the OP $x_0$, then the coefficients $a_0, a_1, a_2, \dots, a_m, \dots$ are given by

$$\boxed{a_m = \frac{y^{(m)}(x_0)}{m!} \quad , \; m = 0, 1, 2, \dots}$$

Answer:

$$a_0 = \frac{y(x_0)}{0!} = y_0 \quad ✓$$

$$a_1 = \frac{y'(x_0)}{1!} = y'_0 \quad ✓$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \Rightarrow y' = \sum_{n=1}^{\infty} n\, a_n (x - x_0)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n (x - x_0)^{n-2}$$

$$\vdots$$

$$y^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)(n-2)\cdots(n-(m-1))\, a_n (x - x_0)^{n-m}$$

$$y^{(m)}(x) = m(m-1)(m-2)\cdots(m-m+1)\,a_m + \sum_{n=m+1}^{\infty} n(n-1)(n-2)\cdots(n-(m-1))\,a_n\,(x-x_0)^{n-m}$$

$$y^{(m)}(x_0) = m!\,a_m + 0 \quad \Rightarrow \quad a_m = \frac{y^{(m)}(x_0)}{m!}$$

---

**Exp** Given the IVP:

$$(x+1)y'' - \ln(e+x^2)\,y' - 2y = 0, \quad y(0)=1, \; y'(0)=1$$

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is the series solution of this IVP, find the first four terms.

$x_0 = 0$ is OP since $P(0) = 1 \neq 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$a_0 = y_0 = 1 \qquad a_1 = y'_0 = 1 \qquad \frac{3}{2} \qquad \frac{1}{3}$

$$a_2 = \frac{y''(x_0)}{2!} = \frac{y''(0)}{2} = \frac{3}{2}$$

$$a_3 = \frac{y'''(x_0)}{3!} = \frac{y'''(0)}{6} = \frac{2}{6} = \frac{1}{3}$$

$$(x+1)y'' - \ln(e+x^2)\,y' - 2y = 0$$
$$(0+1)\,y''(0) - \ln(e+0^2)\,y'(0) - 2y(0) = 0$$
$$y''(0) - \ln e\; y'(0) - 2y(0) = 0$$
$$y''(0) - (1)(1) - 2(1) = 0$$
$$\boxed{y''(0) = 3}$$

To find $y'''(0) \Rightarrow$

$$(x+1)y''' + y'' - \ln(e+x^2)\,y'' - \frac{2x}{e+x^2}\,y' - 2y' = 0$$
$$y'''(0) + y''(0) - \ln e\; y''(0) - 0 - 2y'(0) = 0$$
$$y'''(0) + 3 - (1)(3) - 2(1) = 0 \Rightarrow \boxed{y'''(0) = 2}$$

The 1st four terms: $\{1, x, \frac{3}{2}x^2, \frac{1}{3}x^3\}$

Exp Given the IVP:

$$y'' + xy' + y = 0 \quad , y(0) = 1, \ y'(0) = 0$$

[1] Suppose $y = \phi(x)$ is solution to this IVP.
Find $\phi''(0)$, $\phi'''(0)$, $\phi^{(4)}(0)$

- $y''(0) + (0)\, y'(0) + y(0) = 0$
  $y''(0) + 0 + 1 = 0 \quad \Rightarrow y''(0) = \phi''(0) = -1$

- To find $\phi'''(0)$ we derive $\Rightarrow y''' + xy'' + y' + y' = 0$
  $y'''(0) + 0 + 2y'(0) = 0$
  $y'''(0) = \phi'''(0) = -2y'_0 = 0$

- To find $\phi^{(4)}(0)$ we derive $\Rightarrow y'''' + xy''' + y'' + 2y'' = 0$
  $y^{(4)}(0) + 0 + 3y''(0) = 0$
  $y^{(4)}(0) = \phi^{(4)}(0) = -3y''(0) = 3$

---

[2] Find the 1st three nonzero terms of the power series solution about $x_0 = 0$

- $x_0 = 0$ is OP since $P(x) = 1$ never zero

- The power series solution is $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$a_0 = y_0 = 1$ and $a_1 = y'_0 = 0$

$a_2 = \frac{y''(0)}{2!} = \frac{-1}{2}$ and $a_3 = \frac{y'''(0)}{3!} = 0$

$a_4 = \frac{y^{(4)}(0)}{4!} = \frac{3}{24} = \frac{1}{8}$

$y(x) = 1 + 0 - \frac{1}{2}x^2 + 0 + \frac{x^4}{8} + \cdots$

Hence, the 1st three nonzero terms
$\left\{1, -\frac{1}{2}x^2, \frac{x^4}{8}\right\}$

Exp solve the IVP :

$$y'' - 4e^{2x}y' - (3x^2 + 2x + 5)y = 0 \quad , \quad y(0) = y_0$$
$$y'(0) = y_0'$$

- $P(x) = 1$ never zero $\Rightarrow$ all points are ordinary
  $\Rightarrow x_0 = 0$ is OP

- The series solution about the OP $x_0 = 0$ is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$a_0 \to y_0 \checkmark, \quad a_1 \to y_0' \checkmark, \quad a_2 \to \frac{y_0''}{2!}, \quad a_3 \to \frac{y_0'''}{3!}, \quad a_4 \to \frac{y_0^{(4)}}{4!}$$

- To find $a_2 \Rightarrow y''(0) - 4e^0 y'(0) - (5)y(0) = 0$

$$y''(0) - 4(1)y_0' - 5y_0 = 0$$

$$y''(0) - 4a_1 - 5a_0 = 0 \Rightarrow y''(0) = 4a_1 + 5a_0$$

$$a_2 = \frac{y''(0)}{2!} = \frac{4a_1 + 5a_0}{2} = 2a_1 + \frac{5}{2}a_0$$

- To find $a_3 \Rightarrow y''' - 4e^{2x}y'' - 8e^{2x}y' - (3x^2 + 2x + 5)y' - (6x + 2)y = 0$

$$y'''(0) - 4y''(0) - 8y'(0) - (5)y'(0) - (2)y(0) = 0$$

$$y'''(0) - 4(4a_1 + 5a_0) - 8a_1 - 5a_1 - 2a_0 = 0$$

$$y'''(0) - 29a_1 - 22a_0 = 0$$

$$\Rightarrow y'''(0) = 29a_1 + 22a_0$$

$$a_3 = \frac{y'''(0)}{3!} = \frac{29a_1 + 22a_0}{6} = \frac{29}{6}a_1 + \frac{11}{3}a_0$$

$$y(x) = a_0 + a_1 x + (2a_1 + \tfrac{5}{2}a_0)x^2 + (\tfrac{29}{6}a_1 + \tfrac{11}{3}a_0)x^3 + \cdots$$

$$= a_0\left(1 + \tfrac{5}{2}x^2 + \tfrac{11}{3}x^3 + \cdots\right) + a_1\left(x + 2x^2 + \tfrac{29}{6}x^3 + \cdots\right)$$

$$= a_0\, y_1(x) + a_1\, y_2(x)$$

Th (5.3.1)

If $x_0$ is an OP for the DE

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad --- \; *$$

where

$$p(x) = \frac{Q(x)}{P(x)} \quad \text{and} \quad q(x) = \frac{R(x)}{P(x)} \quad \text{are analytic at } x_0,$$

then the general solution of the DE * is given by

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0\, y_1(x) + a_1\, y_2(x)$$

where

- $a_0$ and $a_1$ are arbitrary constants
- $y_1$ and $y_2$ are two power series solution which are analytic at $x_0$
- the series solutions $y_1$ and $y_2$ form fundemental set of solution

Furthermore, the radius of convergence for each power series solution $y_1(x)$ and $y_2(x)$ is given by

$$\rho = \min\{\rho_1, \rho_2\}$$

where

$\rho_1$ is the radius of convergence for the power series of $p(x)$

and

$\rho_2$ is the radius of convergence for the power series of $q(x)$

---

Remark[1] Th (5.3.1) provides strategy to find $\rho$ for power series solution $y(x) = \sum a_n (x - x_0)^n$ for a given DE about OP $x_0$ without solving the DE

**Remark²** ① If $P(x), Q(x), R(x)$ are all poly. then we can find $\rho_1$ and $\rho_2$ straight forward for $p(x)$ and $q(x)$.

② If $P(x), Q(x), R(x)$ are not all poly. then first we find Taylor series for $p(x)$ and $q(x)$ then find $\rho_1$ and $\rho_2$

---

**Exp** Determine a lower bound for the radius of convergence $\rho$ of the series solution of

① $y'' - xy = 0$ about $x_0 = 1$

$\left.\begin{aligned} P(x) &= 1 \\ Q(x) &= 0 \\ R(x) &= -x \end{aligned}\right\} \Rightarrow$ all poly.

$P(x) = 1$ never zero
all points are ordinary
$x_0 = 1$ is OP

$p(x) = \dfrac{Q(x)}{P(x)} = \dfrac{0}{1} = 0$ is analytic everywhere $\Rightarrow \rho_1 = \infty$

$q(x) = \dfrac{R(x)}{P(x)} = \dfrac{-x}{1} = -x$ is analytic everywhere $\Rightarrow \rho_2 = \infty$

Hence, the radius of convergence $\rho$ for the series solution $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ is $\min\{\rho_1, \rho_2\} = \infty$

by Th 5.3.1

---

② $(x^2+3x)y'' + y' + y = 0$ about $x_0 = -1$

$P(x) = x^2+3x$
$Q(x) = 1$
$R(x) = 1$
} All poly.

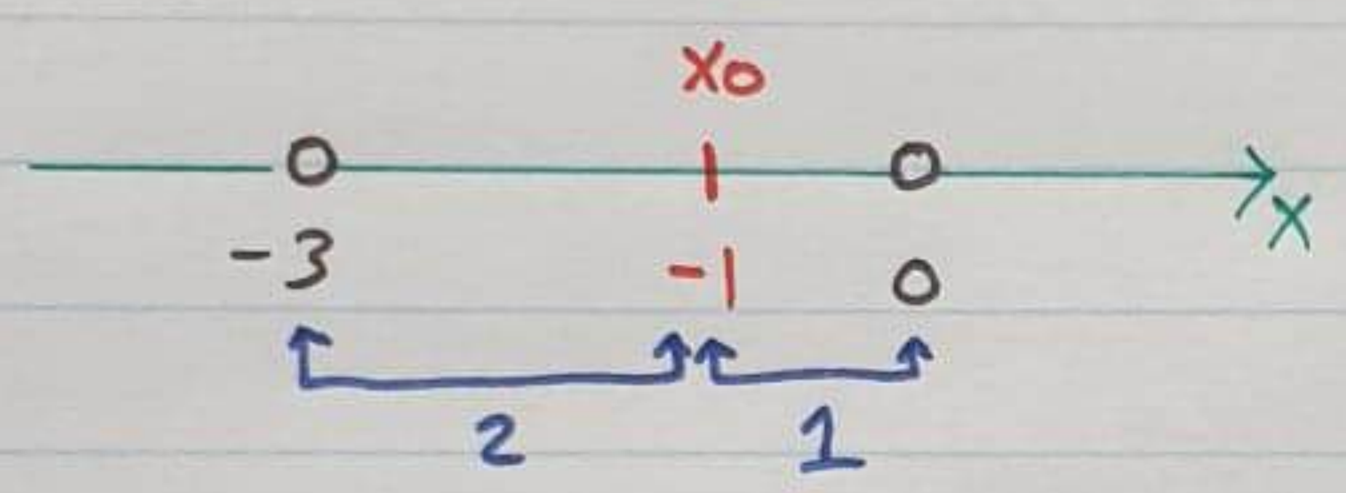
$P(x) = x(x+3) = 0$
$x=0,\ x=-3$
Singlur Points
$x_0 = -1$ is OP

$p(x) = \dfrac{Q(x)}{P(x)} = \dfrac{1}{x(x+3)}$

$q(x) = \dfrac{R(x)}{P(x)} = \dfrac{1}{x(x+3)}$

$\Rightarrow$ are analytic everywhere except at $x=0$ and $x=-3$

p(x)




$\Rightarrow \rho_1 = 1$ for $p(x)$

$\rho_2 = 1$ for $q(x)$

Hence, the radius of convergence for the series solution $y(x) = \sum_{n=0}^{\infty} a_n (x+1)^n$ is $\rho = \min\{\rho_1, \rho_2\} = 1$ by Th5.3.1

---

③ $(1+x^2)y'' + 2xy' + 4x^2y = 0$ about $x_0 = 0$
$x_0 = \frac{1}{2}$

$P(x) = 1+x^2$
$Q(x) = 2x$
$R(x) = 4x^2$
} All poly.

$P(x) = 1+x^2 = 0$
$x = \pm i$
Singular Points
$x_0 = 0$ and $x_0 = \frac{1}{2}$ are OP

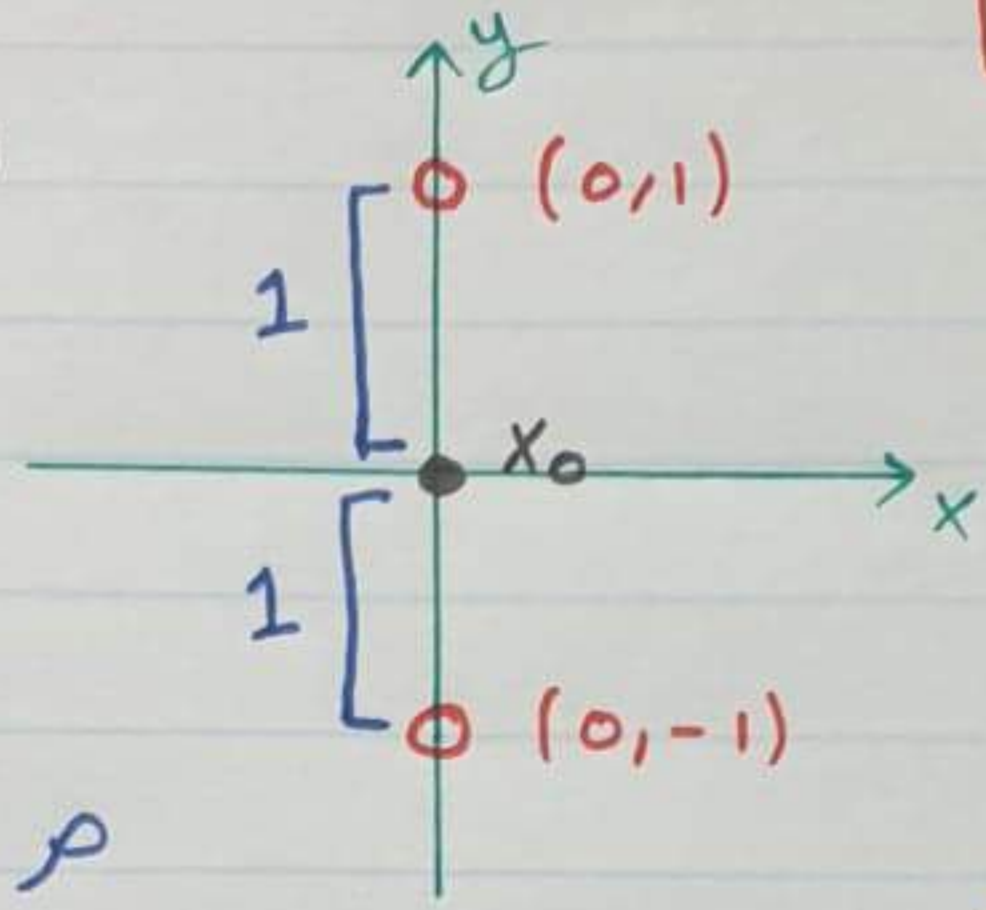
$p(x) = \dfrac{2x}{1+x^2}$

$q(x) = \dfrac{4x^2}{1+x^2}$

$\Rightarrow$ are analytic everywhere except at $x = \pm i = 0 \pm i$ comparing with $z = x+yi$

$(0,1)$ or $(0,-1)$

$\boxed{x_0 = 0}$

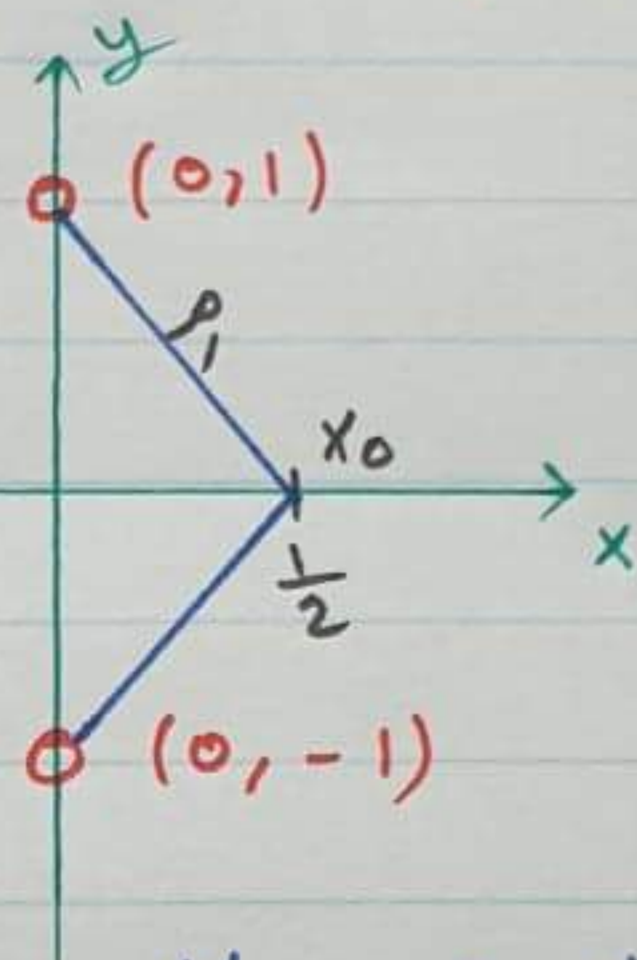



$\rho_1 = 1$ for $p(x)$

$\rho_2 = 1$ for $q(x)$ "similar"

Hence, the radius of convergence $\rho$ for the series solution $\sum a_n x^n$ is $\rho = \min\{1,1\} = 1$

$\boxed{x_0 = \frac{1}{2}}$




$\rho_1 = \sqrt{\left(\frac{1}{2}\right)^2 + (1)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$ for $p(x)$

$\rho_2 = \frac{\sqrt{5}}{2}$ for $q(x)$ "similar"

Hence, the radius of convergence $\rho$ for the series solution $\sum a_n (x - \frac{1}{2})^n$ is $\min\{\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\} = \frac{\sqrt{5}}{2}$

---

(4) $(1+x^2)y'' + (1+x^2)y' + y = 0$ about $x_0 = 0$

$\left.\begin{aligned} P(x) &= 1+x^2 \\ Q(x) &= 1+x^2 \\ R(x) &= 1 \end{aligned}\right\}$ all poly

$P(x) = 0 \Rightarrow 1+x^2 = 0 \Rightarrow$
$x = \pm i$ Singular Points

$x_0 = 0$ is OP

$p(x) = 1$ is analytic everywhere $\Rightarrow \rho_1 = \infty$

$q(x) = \frac{1}{1+x^2}$ is analytic everywhere except at $x = \pm i = 0 \pm i$
(0,1), (0,-1)

$\Rightarrow \rho_2 = 1$ by part (3)

Hence, the radius of convergence for the series solution $\sum a_n x^n$ is $\rho = \min\{1, \infty\} = 1$

⑤ $x(x^2-2x+2)y'' + xy' + (x^2-2x+2)y = 0$ about $x_0 = 2$

$$\left.\begin{aligned} P(x) &= x(x^2-2x+2) \\ Q(x) &= x \\ R(x) &= x^2-2x+2 \end{aligned}\right\} \text{All poly}$$

$P(x) = 0$

$x(x^2-2x+2) = 0$

$x = 0, \quad x = \dfrac{2 \pm \sqrt{4-4(2)}}{2}$

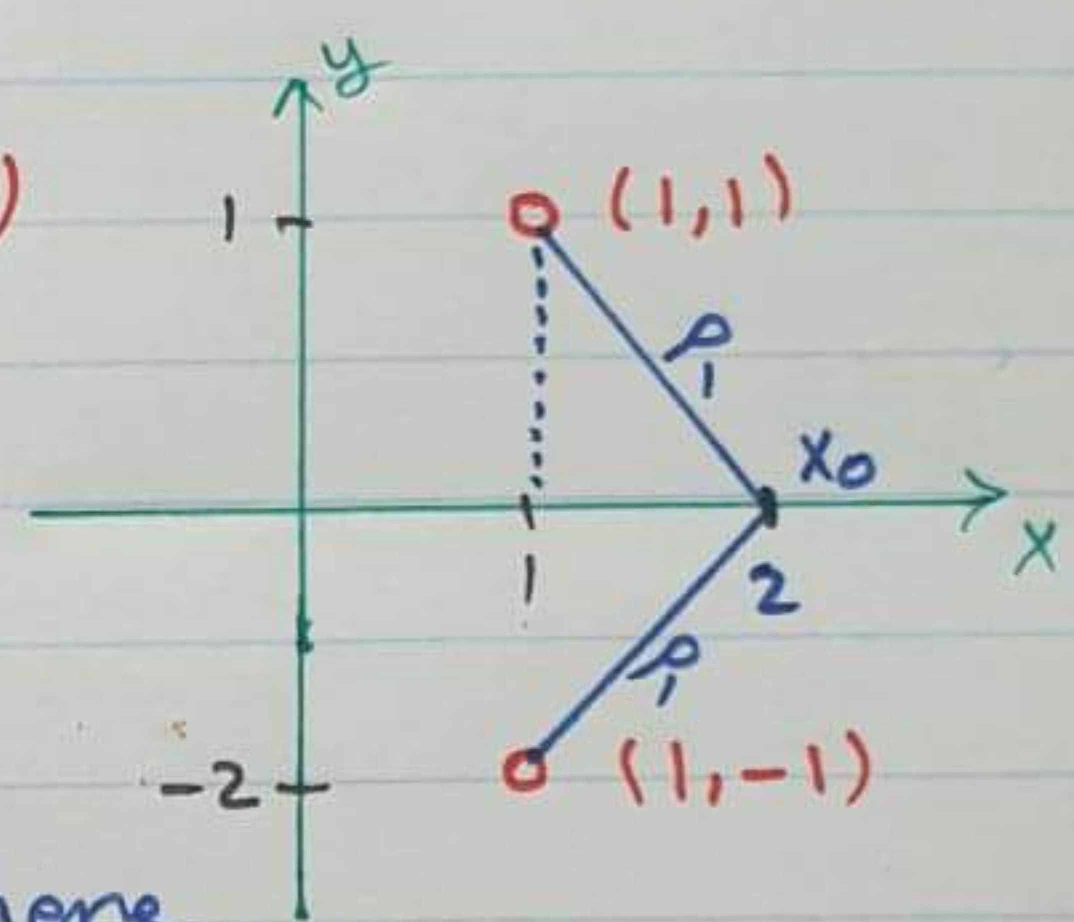
$= 1 \pm i$

$x_0 = 2$ is OP

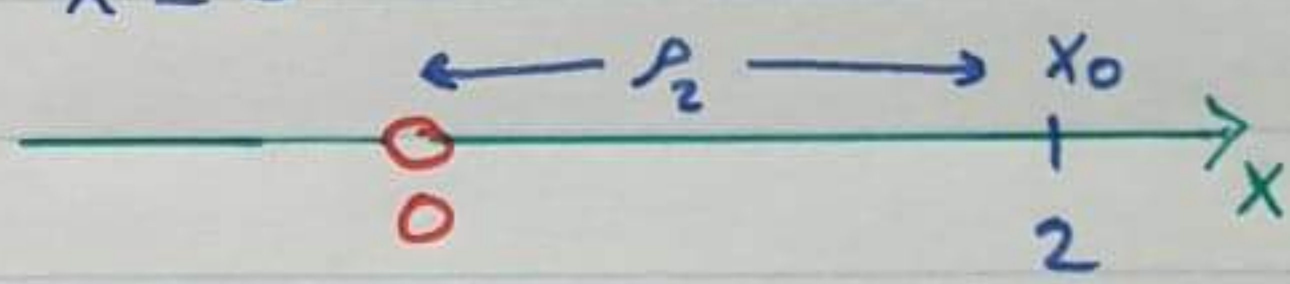
$p(x) = \dfrac{1}{x^2-2x+2}$ is analytic every where except at $x = 1 \pm i$

$(1,1), (1,-1)$

$\rho_1 = \sqrt{1^2+1^2} = \sqrt{2}$ for $p(x)$




$q(x) = \dfrac{1}{x}$ is analytic everywhere except at $x = 0$




$\rho_2 = 2$ for $q(x)$

Hence, the radius of convergence for the series solution $\sum a_n (x-2)^n$ is

$$\rho = \min\{\rho_1, \rho_2\} = \min\{\sqrt{2}, 2\} = \sqrt{2}$$

---

Now we will consider an example when $P(x)$, $Q(x)$, $R(x)$ are not all poly.

⑥ $y'' + (\sin x)\, y' + (1+x^2)\, y = 0$ about $x_0 = 0$

$P(x) = 1$
$Q(x) = \sin x$
$R(x) = 1 + x^2$
$\Big\}$ Not all Poly.

$P(x) = 1$ never zero
$\Rightarrow$ all points are ordinary
$\Rightarrow x_0 = 0$ is OP

$p(x) = \frac{\sin x}{1} = \sin x$ which is analytic every where $\Rightarrow \rho_1 = \infty$

$q(x) = \frac{1+x^2}{1} = 1 + x^2$ which is analytic every where $\Rightarrow \rho_2 = \infty$

Hence, the radius of convergence $\rho$ for the series solution $\sum a_n x^n$ is $\infty$

---

Basically we find Taylor series expansion for $\sin x$ about $x_0 = 0$
" Maclurine Series " $\Rightarrow$

$$p(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} \Rightarrow \text{Apply RT}$$

$$\Rightarrow \rho_1 = \infty$$

same for $q(x) = 1 + x^2$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

$$= (1) + (0)x + \frac{2}{2}x^2 + 0 + 0 + \cdots$$

$$= 1 + x^2$$

$f(x) = 1 + x^2$
$f'(x) = 2x$
$f''(x) = 2$
$f'''(x) = 0$
$f^{(4)}(x) = 0$
$\vdots$

⑦ $(x^2+1)y'' + xy' + \frac{1}{x-2}y = 0$ about $x_0 = 1$

Multiply all terms by $x-2$

$$(x-2)(x^2+1)y'' + x(x-2)y' + y = 0$$

$\left.\begin{array}{l} P(x) = (x-2)(x^2+1) \\ Q(x) = (x-2)x \\ R(x) = 1 \end{array}\right\}$ All poly.

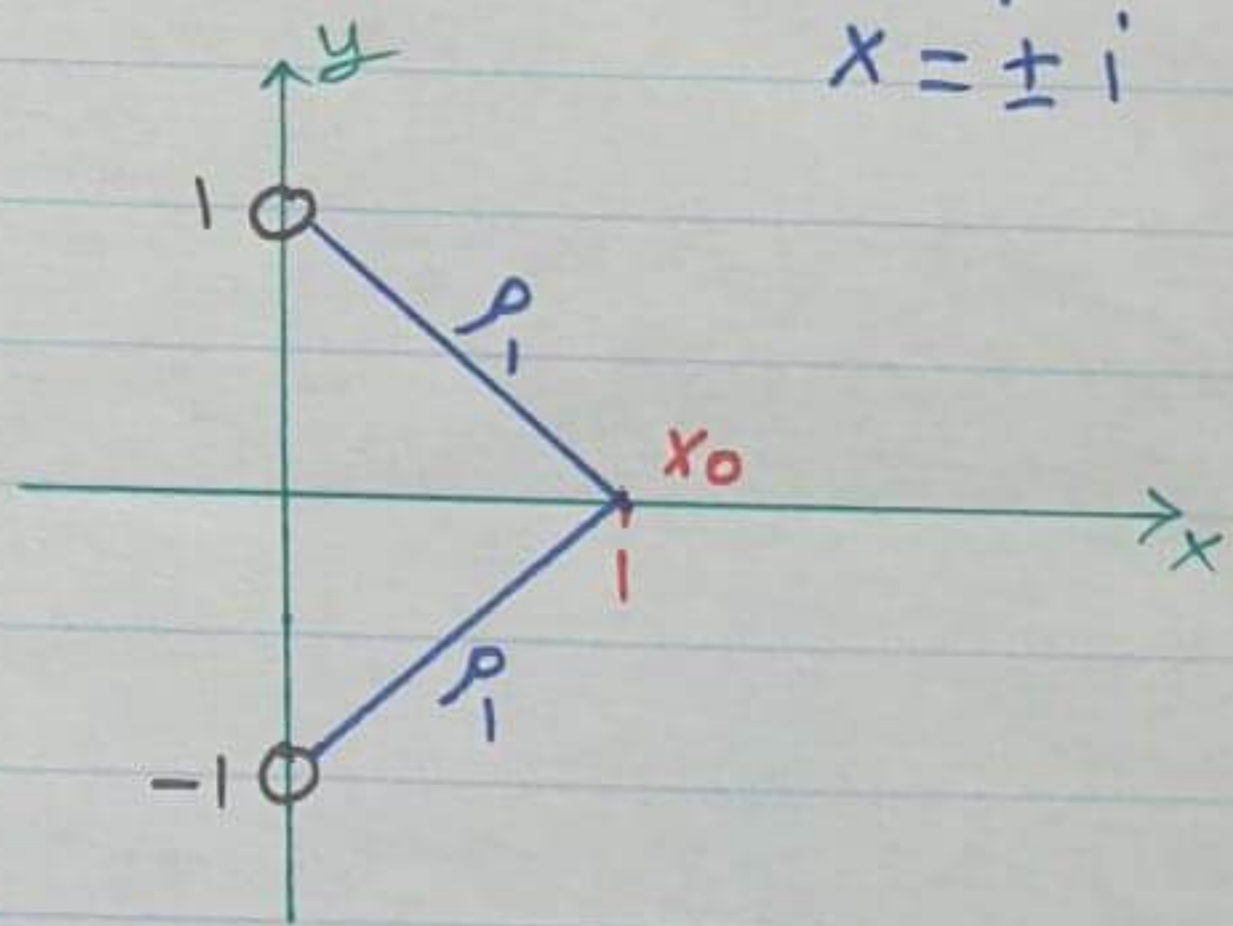
$P(x) = 0 \iff (x-2)(x^2+1) = 0 \iff x = 2,\ x = \pm i$

Singular Points

$\Rightarrow$ $x_0 = 1$ is OP

$(0,1), (0,-1)$

$p(x) = \frac{Q(x)}{P(x)} = \frac{x}{x^2+1}$ is analytic every where except at $x = \pm i$

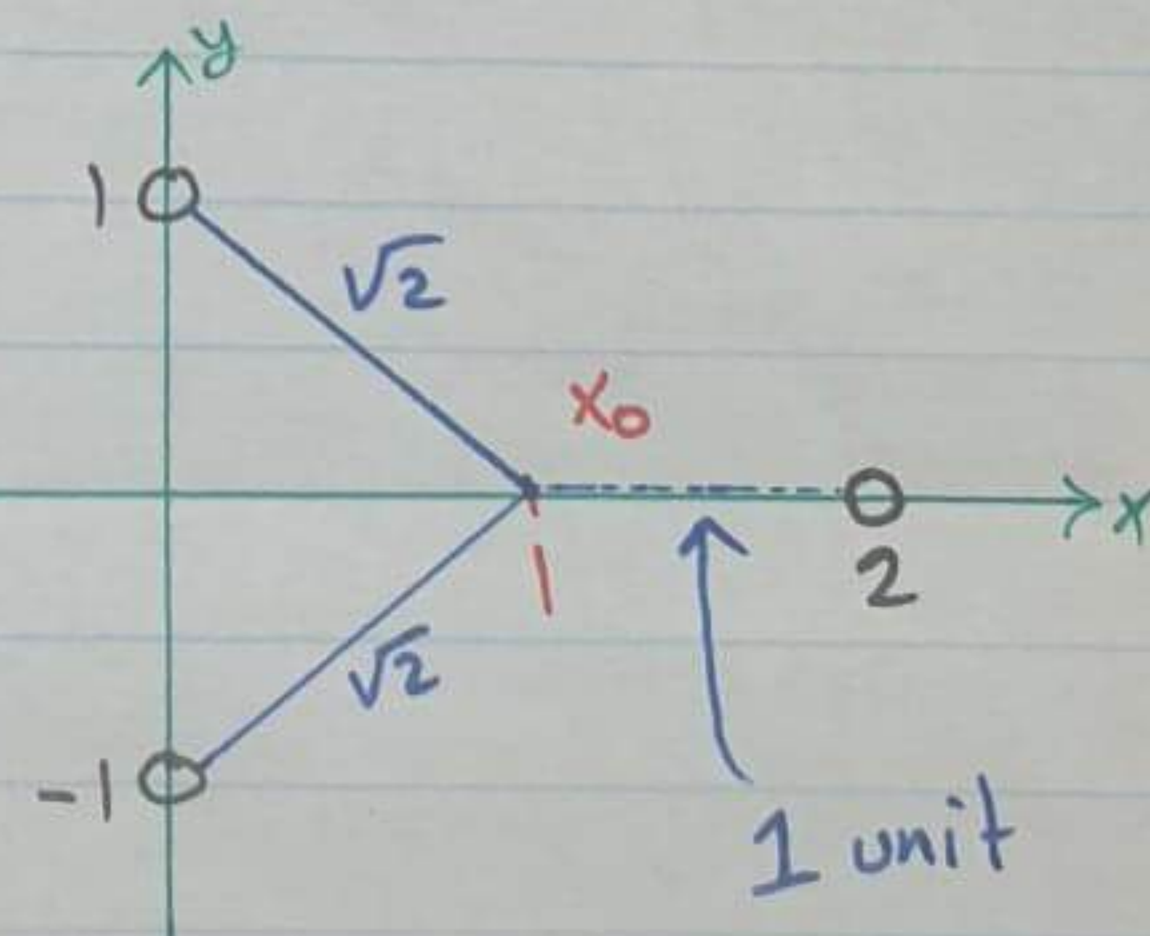



$\rho_1 = \sqrt{1^2+1^2} = \sqrt{2}$

$q(x) = \frac{R(x)}{P(x)} = \frac{1}{(x-2)(x^2+1)}$

is analytic every where except $x = 2$ and $x = \pm i$

$\rho_2 = \min\{1, \sqrt{2}\} = 1$




• Hence, the radius of convergence for the power series solution $\sum a_n(x-1)^n$ is $\rho = \min\{\rho_1, \rho_2\}$

$= \min\{\sqrt{2}, 1\}$

$= 1$

# 5.4 Euler Differential Equations ; Regular Singular Points

- Recall the Euler DE : $x^2 y'' + \alpha x y' + \beta y = 0$

- Note that Euler DE has a singular point at $x_0 = 0$ since $P(x) = x^2 \Rightarrow P(x) = 0 \Leftrightarrow x^2 = 0 \Leftrightarrow x_0 = 0$

- If we try to find a power series solution for Euler DE about the SP $x_0 = 0$ " $y(x) = \sum_{n=0}^{\infty} a_n x^n$ " as in 5.2 , then we will find out that this is impossible.

The reason for that is due to the fact that $p(x)$ and $q(x)$ are not analytic at the SP $x_0 = 0$

- Thus, we need more information about the singularity of $p(x)$ and $q(x)$ to be not too severve

- So first we will classify the SP's into

Regular Singular Point (RSP) or
Irregular Singular Point (IRSP)

- In section 5.5 we will find power series solution in the neighborhood of a $\underset{x_0 = 0}{\underbrace{\text{RSP}}}$ for a given DE.

- Given a $2^{nd}$ order linear DE :

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0 \quad \cdots \quad *$$

- Assume the DE * has a SP at $x_0$ $(P(x_0) = 0)$ :

$\boxed{1}$ If $P, Q, R$ are all poly., then $x_0$ is RSP if

$$\lim_{x \to x_0} (x - x_0)\, p(x) < \infty \quad \text{and}$$

$$\lim_{x \to x_0} (x - x_0)^2\, q(x) < \infty$$

$\boxed{2}$ If $P, Q, R$ are functions more general than poly., then $x_0$ is RSP if

$$(x - x_0)\, p(x) \quad \text{and} \quad (x - x_0)^2\, q(x)$$

are analytic about $x_0$ (They have Taylor Series Expansion about $x_0$ with $\rho$ s.t $|x - x_0| < \rho$)

---

Remark: If the Singular point $x_0$ is not regular, then $x_0$ is $\underbrace{\text{IRSP}}$.

Irregular Singular Point

**Exp** Determine the singular points of the following DE's and classify them into RSP or IRSP :

① $x^2 y'' + \alpha x y' + \beta y = 0$, $\alpha$ and $\beta$ constants "Euler DE"

$$\left.\begin{aligned} P(x) &= x^2 \\ Q(x) &= \alpha x \\ R(x) &= \beta \end{aligned}\right\} \text{All poly.} \quad \Rightarrow P(x) = 0 \iff \boxed{x_0 = 0} \text{ is SP}$$

Apply [I]

$$\lim_{x \to x_0} (x - x_0) p(x) = \lim_{x \to 0} x \frac{\alpha x}{x^2} = \alpha < \infty \checkmark \quad \text{and}$$

$$\lim_{x \to x_0} (x - x_0)^2 q(x) = \lim_{x \to 0} x^2 \frac{\beta}{x^2} = \beta < \infty \checkmark$$

Hence, $x_0 = 0$ is RSP and so any Euler DE has a RSP at $x_0 = 0$

---

② $(1 - x) y'' - 2x y' + 4y = 0$

$$\left.\begin{aligned} P(x) &= 1 - x \\ Q(x) &= -2x \\ R(x) &= 4 \end{aligned}\right\} \text{All poly.} \qquad P(x) = 0 \iff 1 - x = 0 \iff \boxed{x_0 = 1} \text{ is SP}$$

Apply [I] $\Rightarrow \displaystyle\lim_{x \to x_0} (x - x_0) p(x) = \lim_{x \to 1} (x - 1) \frac{-2x}{(1 - x)} = \lim_{x \to 1} 2x = 2 < \infty$

$$\lim_{x \to x_0} (x - x_0)^2 q(x) = \lim_{x \to 1} (x - 1)^2 \frac{4}{(1 - x)} = \lim_{x \to 1} -4(x - 1) = 0 < \infty$$

Hence, $x_0 = 1$ is RSP

③ $2x(x-2)^2 y'' + 3x y' + (x-2) y = 0$

$P(x) = 2x(x-2)^2$
$Q(x) = 3x$
$R(x) = (x-2)$
All poly.

$P(x) = 0$
$2x(x-2) = 0$
$x_0 = 0$ and $x_0 = 2$ are SP's.

Apply ①

$\boxed{x_0 = 0}$

$$\lim_{x \to x_0} (x - x_0) p(x) = \lim_{x \to 0} x \frac{3x}{2x(x-2)^2} = \frac{3}{2} \lim_{x \to 0} \frac{x}{(x-2)^2} = 0 < \infty$$

$$\lim_{x \to x_0} (x - x_0)^2 q(x) = \lim_{x \to 0} x^2 \frac{(x-2)}{2x(x-2)^2} = \frac{1}{2} \lim_{x \to 0} \frac{x}{x-2} = 0 < \infty$$

Hence, $x_0 = 0$ is RSP

$\boxed{x_0 = 2}$

$$\lim_{x \to x_0} (x - x_0) p(x) = \lim_{x \to 2} (x-2) \frac{3x}{2x(x-2)^2} = \frac{3}{2} \lim_{x \to 2} \frac{1}{x-2} \quad \text{DNE}$$

Hence, $x_0 = 2$ is IRSP.

since $$\lim_{x \to 2^+} \frac{1}{x-2} = \frac{1}{\text{small} +} = \infty$$

$$\lim_{x \to 2^-} \frac{1}{x-2} = \frac{1}{\text{small} -} = -\infty$$

④ $(x-\frac{\pi}{2})^2 y'' + \cos x \, y' + \sin x \, y = 0$

$P(x) = (x-\frac{\pi}{2})^2$
$Q(x) = \cos x$
$R(x) = \sin x$
} Not All Poly.

$\Lleftarrow$ Apply $\boxed{2}$

$P(x) = 0$
$(x-\frac{\pi}{2})^2 = 0$
$\boxed{x_0 = \frac{\pi}{2}}$ is SP

$(x-x_0)\,p(x) = (x-\frac{\pi}{2})\dfrac{\cos x}{(x-\frac{\pi}{2})^2} = \dfrac{\cos x}{x-\frac{\pi}{2}}$

$(x-x_0)^2 q(x) = (x-\frac{\pi}{2})^2 \dfrac{\sin x}{(x-\frac{\pi}{2})^2} = \sin x$

} we find Taylor Series Expansion about $x_0 = \frac{\pi}{2}$

First we find Taylor series for $f(x) = \cos x$ about $x_0 = \frac{\pi}{2}$

$f(x) = \cos x \Rightarrow f(\frac{\pi}{2}) = 0$
$f'(x) = -\sin x \Rightarrow f'(\frac{\pi}{2}) = -1$
$f''(x) = -\cos x \Rightarrow f''(\frac{\pi}{2}) = 0$
$f'''(x) = \sin x \Rightarrow f'''(\frac{\pi}{2}) = 1$
$f^{(4)}(x) = \cos x \Rightarrow f^{(4)}(\frac{\pi}{2}) = 0$
$\vdots$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{2})}{n!}(x-\frac{\pi}{2})^n$$

$$= f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x-\frac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x-\frac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x-\frac{\pi}{2})^3 + \cdots$$

$$= 0 + (-1)(x-\frac{\pi}{2}) + 0 + \frac{1}{3!}(x-\frac{\pi}{2})^3 + 0 + \cdots$$

$$= -(x-\frac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{3!} - \frac{(x-\frac{\pi}{2})^5}{5!} + \frac{(x-\frac{\pi}{2})^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$$

Hence, the Taylor series expansion for

$$\frac{\cos x}{x-\frac{\pi}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (x-\frac{\pi}{2})^{2n}}{(2n+1)!} = -1 + \frac{(x-\frac{\pi}{2})^2}{3!} - \frac{(x-\frac{\pi}{2})^4}{5!} + \cdots$$

Second we find Taylor series expansion for $g(x) = \sin x$ about $x_0 = \frac{\pi}{2}$

$g(x) = \sin x \Rightarrow g(\frac{\pi}{2}) = 1$
$g'(x) = \cos x \Rightarrow g'(\frac{\pi}{2}) = 0$
$g''(x) = -\sin x \Rightarrow g''(\frac{\pi}{2}) = -1$
$g'''(x) = -\cos x \Rightarrow g'''(\frac{\pi}{2}) = 0$
$g^{(4)}(x) = \sin x \Rightarrow g^{(4)}(\frac{\pi}{2}) = 1$
$\vdots$

$$\sin x = \sum_{n=0}^{\infty} \frac{g^{(n)}(\frac{\pi}{2})}{n!} (x-\frac{\pi}{2})^n = g(\frac{\pi}{2}) + g'(\frac{\pi}{2})(x-\frac{\pi}{2}) + \frac{g''(\frac{\pi}{2})}{2!}(x-\frac{\pi}{2})^2 + \cdots$$

$$= 1 + 0 - \frac{1}{2!}(x-\frac{\pi}{2})^2 + 0 + \frac{1}{4}(x-\frac{\pi}{2})^4 + 0 + \cdots$$

$$= 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \frac{(x-\frac{\pi}{2})^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-\frac{\pi}{2})^{2n}}{(2n)!}$$

Hence, $x_0 = \frac{\pi}{2}$ is RSP