بسم الله الرحمن الرحيم

Mid revision

تلخيص المطلوب من كل شابتر في مادة الميد (قوانين ، أمثلة ، ملاحظات ...)

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1.3

1- verify that y is a Solution $\Rightarrow$ Substitute y

2- See if it's linear or not / find the order

1.1 generate the differential equation

Free fall : $\frac{dv}{dt} = g - \frac{\alpha}{m} v$

- Find the behavior :-

1. نساوي المشتقة بالصفر
2. نوجد عندما $y=0$
3. نرسم
4. ناخذ قيمة اكبر من y وقيمة اصغر ونعوض في y'
5. if $y' > 0$: y diverges (goes to ∞ or $-\infty$) $\uparrow$
   if $y' < 0$: y Converges (goes to y_0) $\downarrow$
6. Find $\lim_{t \to \infty} y(t)$ (i can take it from the Plot)

Behavior : $\lim y(t) = \begin{cases} \infty, & y_0 > X \\ X, & y_0 = X \\ -\infty, & y_0 < X \end{cases}$

- How to generate the D.E in these questions ?

$\frac{dy}{dx} = \text{Rate in} - \text{Rate out}$

2.2

① separable

make dy in one side and dt in the other side then integrate
Sometimes you need to assume u or use Partial fraction for integration

② Homogenous

general form : $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ or $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$

how to solve ?

① let $y = vx$ ② $\frac{dy}{dx} = v + x\frac{dv}{x}$ ③ Substitute then solve

③ Liner (method of integrating factor)

general form : $y' + P(t)y = g(t)$

how to solve ?

$y(t) = \frac{1}{\mu t}\int (\mu t\, g(t)\, dt + c)$, $\mu_t = e^{\int P(t)\,dt}$

2.3 Tank + Cooling

1- Tank

① Rate in − Rate out

= (flow in * التركيز) − (flow out * التركيز)

* Sometimes you have to build an equation for تركيز
بناءً على كمية المياه الداخلة او الخارجة وسعة الخزانات

* التركيز = كمية الملح / كمية المياه * دايما بعوض Q مكان كمية الملح

⇐ التركيز = Q / كمية المياه

2- Newton's Law of Cooling

$$\frac{du}{dt} = -K(u-T)$$

u : The temperature of the object
T : The temperature of the room
K : Positive Constant

1. divide by (u − t) then integrate
2. the answer will be ln| something| , give it e

2.4 See if there is a Unique Solution Without Solving + Bernoulli

1- Linear D.E

1. make it like the form $y' + P(t)y = g(t)$
2. نوجد أصفار المقام
3. Find the interval that Contains the initial Value t_0
4. if we did all of this $\Rightarrow$ There is a Solution and it's Unique

* if he tells me to find this Solution i have to solve the IVP

2- Non-Linear D.E

i have to find an interval in y that contains y_0 , $y_0 \in (y_1, y_2)$
and an interval in t that contains t_0 , $t_0 \in (t_1, t_2)$

1. differentiate y' to get y''
2. See the interval of Both y' and y'' where they are defined
3. Check if their interval Contains (y_0, t_0) , Yes $\Rightarrow$ have Unique Sol.
   No, we have to solve the original IVP

3- Bernoulli

general form : $y' + P(t)y = g(t)y^n$, $n \in R$

if $n=0$:-

$y' + P(t)y = g(t) \Rightarrow$ Linear

if $n=1$:-

$y' + (g(t) - P(t))y = 0 \Rightarrow$ seprable or linear

if $n \neq 0, 1$:-

Let $V = y^{1-n}$, $V' + P(t)(1-n)V = (1-n)g(t)$
and Solve it by Linear in V, then Substitute $V = y^{1-n}$

2.6 Exact equations

① if it's exact

$2x + y^2 + 2xyy' = 0 \quad , \quad y(1) = 1$

① Find M and N

$N = 2xy$

$M = 2x + y^2$

② Differentiate N By respect to X, and M By respect to y

$N_x = 2y$

$M_y = 2y$

They are equal $\Rightarrow$ exact

③ Choose N or M to integrate "does not matter"

- Note :- $f_x = M$ and $f_y = N$

$f = \int f_y = \int N \Rightarrow f = \int 2xy\, dy = xy^2 + g(x)$

④ To get $g(x)$ you need to Solve with the other one

1- Set f_x = other one $\Rightarrow f_x = M$

2- differentiate with respect to other one

$f_x = y^2 + g'(x) = M \Rightarrow \cancel{y^2} + g'(x) = 2x + \cancel{y^2}$

$g'(x) = 2x \Rightarrow g(x) = x^2$

⑤ we Put it in it's Place

$xy^2 + x^2 = C$

⑥ Use the Condition to find C

$1 + 1 = C \Rightarrow C = 2$

$\Rightarrow \boxed{xy^2 + x^2 = 2}$

② if it's Not exact

① $\dfrac{M_y - N_x}{N} = f(x) \Rightarrow I = e^{\int f(x)\,dx}$

or

② $\dfrac{M_y - N_x}{M} = f(y) \Rightarrow I = e^{-\int f(y)\,dy}$

then we multiply the equation by I to make it exact
then we start again from the beginning

Euler Formula :- $e^{ix} = \cos x + i \sin x$

Exp. rewrite $e^{1-\frac{\pi}{3}i}$

$= e * e^{i(-\frac{\pi}{3})} = e\left[\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})\right]$

$= e\left[\cos(\frac{\pi}{3}) - i\sin(\frac{\pi}{3})\right]$

$= e\left[0.5 - \frac{\sqrt{3}}{2} i\right]$

$$\frac{\sqrt{\begin{array}{c|ccccc} & 0 & 30 & 45 & 60 & 90 \\ \cos & 4 & 3 & 2 & 1 & 0 \\ \sin & 0 & 1 & 2 & 3 & 4 \end{array}}}{2}$$

* How to find sin and cos without Calculator

$\boxed{2.8}$ Picard's Iteration

general form :- $\frac{dy}{dt} = f(t,y) \Rightarrow y = \int f(t,y)\,dt$

$$y(t) = \phi(t) = \int_0^t f(t, \phi(t))\,dt$$

$\phi_0 = 0$

$\phi_1 = \int_0^t f(t,0)\,dt$

$\phi_2 = \int_0^t f(t,\phi_1)\,dt$

$\vdots$

$\phi_n = \int_0^t f(t,\phi_{n-1})\,dt$

Then i take the Lim for ϕ_n

$\lim_{t\to\infty} \phi_n$ = the answer

its usudly on a shape like

this $\frac{n\,t^n}{n!}$ or something

$\boxed{2.9}$ missing t and missing y :-

1- missing y :-

$V = y'$, $V' = y''$

2- missing t :-

$V = y'$, $y'' = \frac{dv}{dy} V$

Remark :-

$\cosh t = \frac{e^t + e^{-t}}{2}$

$\sinh t = \frac{e^t - e^{-t}}{2}$

How to solve ?

- Solve for v to get y' by integrating v' then solve for y by integrating y'

- if there is any initial conditions i have to use them to find C_1 and C_2

$\boxed{3.1}$ + $\boxed{3.3}$

1- Coefficients are Constants :-

i solve it by aux. equation, and there is three cases

Case 1 : $r_1 \neq r_2$

$$y_1 = e^{r_1 t} \quad , \quad y_2 = e^{r_2 t}$$

Case 2 : $r_1 = r_2 = r$

$$y_1 = e^{rt} \quad , \quad y_2 = t\,e^{rt}$$

Case 3 : $r = \alpha \pm \omega i$ (Complex roots)

$$r_1 = \alpha + \omega i \quad , \quad r_2 = \alpha - \omega i$$

$$y_1 = e^{\alpha t}\cos(\omega t) \quad , \quad y_2 = e^{\alpha t}\sin(\omega t)$$

Complex : $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -\sqrt{1}$, $i^4 = 1$

2- Cooffetients are Not Constants : (Euler eq.)

$$y = x^r \quad , \quad r^2 + (\alpha - 1)r + B = 0$$

α : $p(t)$ معامل

B : $q(t)$ معامل

Case 1 : $r_1 \neq r_2$

$$y_1 = x^{r_1} \quad , \quad y_2 = x^{r_2}$$

Case 2 : $r_1 = r_2 = r$

$$y_1 = x^r \quad , \quad y_2 = \ln x \; x^r$$

Case 3 : $r = \alpha \pm \omega i$ (Complex roots)

$$y_1 = x^{\alpha}\cos(\omega \ln x) \quad , \quad y_2 = x^{\alpha}\sin(\omega \ln x)$$

3.2

1- how to find the largest interval :-

- do it as the form $y'' + p(t)y' + q(t)y = 0$
- نوجد اصفار المقام
- find the largest interval that Contains $y(t_0) = y_0$

2- Principle of SuperPosition :-

- if y_1 and y_2 are solutions for $y'' + p(t)y' + q(t)y = 0$
  then the linear Combination is also a solution $C_1y_1 + C_2y_2 = 0$

3- The Wronskian Solution $W(y_1, y_2)(t)$:-

[1] if i have y_1 and y_2 :-

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

[2] if i don't have y_1 and y_2 (Abel's Theorem)

$$W(y_1, y_2)(t) = C e^{-\int p(t)\, dt}$$

- The Linear Combination is a Solution iff $W(y_1, y_2)(t) \neq 0$
- y_1 and y_2 are Independent iff $W(y_1, y_2)(t) \neq 0$

4- Fundemental set of Solutions :-

Two Conditions Should Occur to Say that $L(y)$ is F.S.S

1. y_1 and y_2 are Solutions to $y'' + p(t)y' + q(t)y = 0$
2. y_1 and y_2 are IndePendent $(W(y_1, y_2)(t) \neq 0)$

5- Finding F.S.S with no y_1 or y_2, with just interval

1. let "$y_1(t_0) = 1$" "$y_1'(t_0) = 0$" "$y_2(t_0) = 0$" "$y_2'(t_0) = 1$" (to get $W \neq 0$)
2. Find the Charastaretrec equation, then generate y
3. Find C_1 and C_2 in the two Cases in Point 1

- if he gave me interval i Can Choose t_0

On Point 4

② check if $y_1 = x$, $y_2 = \sin x$ form a fundamental set of solutions for the d-e

$$(1 - x\cot x)y'' - xy' + y = 0, \quad 0 < x < \pi.$$

① Verfy if y_1 and y_2 are solutions for the equation :-

$y_1 = x$, $y_1' = 1$, $y_1'' = 0$

$(1 - x\cot x) * 0 - x + x = 0 \Rightarrow y_1$ is solution

$y_2 = \sin x$, $y_2' = \cos x$, $y_2'' = -\sin x$

$(1 - x\cot) * -\sin x - x\cos x + \sin x = 0$

$\cancel{-\sin x} + x\sin x\cot x - x\cos x + \cancel{\sin x} = 0$

$\cancel{x}\left(\cancel{\sin x} * \frac{\cos x}{\cancel{\sin x}}\right) - \cos x) = \cancel{0}$

$\cos x - \cos x = 0 \Rightarrow y_1$ and y_2 are solutions #

② Verify that $W(y_1, y_2) \neq 0$

$W(x, \sin x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \neq 0 \quad \forall\ 0 < x < \pi$

$W(x, \sin x)(\pi/2) \neq 0 \Rightarrow$ they are Lenicrly independent

∴ Since they are solutions and L.E

$\Rightarrow \boxed{\{x, \sin x\}}$ form a fundemental set of solutions #

On Point 5

Ex: Find the fundamental set of solutions specified by thm 3.2.5 for $y'' - y = 0$ using $t_0 = 0$.

$r^2 - 1 = 0 \Rightarrow r_1 = 1 \quad , \quad r_2 = -1$

$y = c_1 y_1 + c_2 y_2 \Rightarrow \boxed{y = c_1 e^t + c_2 e^{-t}}$ ⊛

$\Rightarrow y_1 = e^t \quad , \quad y_2 = e^{-t}$

Case 1: $y_1(0) = 1 \quad , \quad y_1'(0) = 0 \quad , \quad y' = c_1 e^t - c_2 e^{-t}$

from ⊛ $c_1 e^0 + c_2 e^0 = 1 \Rightarrow c_1 + c_2 = 1$ ⊛ في t = 1 و y=0 بعوض

$c_1 e^0 - c_2 e^0 = 0 \Rightarrow c_1 - c_2 = 0$ ⊛ في t=0 و y'=0 بعوض

$\Rightarrow c_1 = c_2 = 1/2$

$y_3 = \frac{e^t + e^{-t}}{2} \Rightarrow \boxed{y_3 = \cosh t}$

Case 2: $y_2(0) = 0 \quad , \quad y_2'(0) = 1 \quad , \quad y' = c_1 e^t - c_2 e^{-t}$

$c_1 e^0 + c_2 e^0 = 0 \Rightarrow c_1 + c_2 = 0$

$c_1 e^0 - c_2 e^0 = 1 \Rightarrow c_1 - c_2 = 1$

$\Rightarrow c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{2}$

$y_4 = \frac{e^t - e^{-t}}{2} \Rightarrow \boxed{y_4 = \sinh t}$

$\therefore$ $\boxed{\{\cosh t , \sinh t\}}$ are Fundemental set of Solutions

#

On Abel's theorem

Q34) If y_1 and y_2 are a fundamental set of solutions of $t y'' + 2y' + te^t y = 0$ and if $W(y_1, y_2)(1) = 2$, find $W(y_1, y_2)(5)$.

$$y'' + \frac{2}{t} y' + e^t y = 0 \implies P(t) = \frac{2}{t}$$

$$W(y_1, y_2)(t) = C e^{-\int P(t)dt} = C e^{-\int \frac{2}{t}} = \frac{C}{t^2}$$

$$W(y_1, y_2)(1) = \frac{C}{1} = 2 \implies C = 2$$

$$W(y_1, y_2)(t) = \frac{2}{t^2} \implies \boxed{W(y_1, y_2)(5) = \frac{2}{25}} \#$$

$\boxed{3.4}$ i have y_1 and want to find y_2

1- Reduction of order Method :-

① let $y = v y_1$ ② Find y' and y'' ③ عوضهن في y^*

if i still have v of order 2 :-

④ let $w = v'$ ⑤ Find w' ⑥ عوضهن في y^*

then find w from $\int w'$, then find y from $\int v'$

2- Reduction of order formula :-

$$y_2 = y_1 \int \frac{W(y_1, y_2)(t)}{y_1^2}$$

Solve the example below using this method ?

$$W(y_1, y_2)(t) = C e^{-\int p(t)} \Rightarrow W(y_1, y_2)(t) = C t^6$$

$$y_2 = t^2 \int \frac{c t^{\cancel{6}\,2}}{\cancel{t^4}} \Rightarrow C t^2 * \frac{t^3}{3}$$

$$\boxed{y_2 = t^5}$$

Ex. Given that $y_1 = t^2$ is a solution of

$(*)$ $\boxed{t\,y'' - 6y' + \frac{10}{t}y = 0, \quad t > 0}$. Use the method of reduction of order to find a second solution of the given d-e.

$$y'' - \frac{6}{t}y' + \frac{10}{t^2}y = 0 \quad , \quad \text{let } y = Vt^2$$

$$y' = 2tV + t^2V' \quad , \quad y'' = 2V + 4tV' + t^2V''$$

$$(2V + 4tV' + t^2V'') - \frac{6}{\cancel{t}}(2\cancel{t}V + t^{\cancel{2}}V') + \frac{10}{\cancel{t^2}}(V\cancel{t^2})$$

$$t^2V'' + 4tV' - 6tV' \cancel{+2V} \cancel{-12V} \cancel{+10V}$$

$$t^2V'' - 2tV' = 0 \implies (tV'' - 2V') = 0 \qquad \text{let } w = V'$$
$$w' = V''$$

$$tw' - 2w = 0 \implies w' - \frac{2}{t}w = 0$$

$$\mu(t) = e^{-\int \frac{2}{t}} \implies \mu(t) = t^{-2}$$

$$w(t) = t^2\left[\int 0 * t^{-2}\,dt + c\right] \implies w = ct^2 = V'$$

$$V = \int ct^2 \implies V = \frac{c}{3}t^3 + c \implies V = At^3 + C \qquad A = \frac{c}{3}$$

$$y = Vy_1 \implies y = (At^3 + C)(t^2)$$

$y = At^5 + Ct^2$ Since this is y_1 then $\boxed{y_2 = t^5}$

ملاحظة :- y هنا حل المعادلة الأساسية $c_1y_1 + c_2y_2$

بس y_2 ما تتخربط

$\boxed{3.5}$ + $\boxed{4.3}$ Non-homogeneous d.e

form : $y'' + P(t)y' + q(t)y = g(t)$

Conditions :- $ay'' + by' + cy = g(t)$

1- a , b , c are Constants

2- g(t) is : Constant or Poly or exponential or sin or cos or finite sum / product of them

How to Solve ?

① Find y_h (The Char. equation)

② Find y_p (Particular Solution)

- $g(t) = \text{number} \Rightarrow y_p = t^s * A$
- $g(t) = t \Rightarrow y_p = t^s(At + B \ldots)$
- $g(t) = te^{\alpha t} \Rightarrow y_p = t^s(At + B)e^{\alpha t}$
- $g(t) = t\, e^{\alpha t} \begin{cases} \sin \\ \cos \end{cases} \Rightarrow y_p = t^s[(At + B)e^{\alpha t} \sin + (Dt + F)e^{\alpha t} \cos]$

t^s : $s = 0, 1, 2$ according if it's independent

③ $y = y_h + y_p$

Note :-

if i have something like $g(t) = 3e^{2t} + 2\sin t$

i do $y_1 = g_1$ with $g_1 = 3e^{2t}$

and $y_2 = g_2$ with $g_2 = 2\sin t$

Finally, $y_p = y_1 + y_2$

Ex ③. Solve $\boxed{y'' - 3y' - 4y = 2\sin t}$.

① Find y_h :-

$y'' - 3y' - 4y = 0 \Rightarrow r^2 - 3r - 4 = 0 \Rightarrow r = -1 \ , r = 4$

$y_h = C_1 y_1 + C_2 y_2 \Rightarrow \boxed{y_h = C_1 e^{-t} + C_2 e^{4t}}$

② Find y_p :-

$y'' - 3y' - 4y = 2\sin t \Rightarrow y_p = t^s [A \sin t + B \cos t]$

$t^s = 0$ because they are independent $\Rightarrow \boxed{y_p = A\sin t + B\cos t}$

نعوض في المعادلة الأصلية لنجيب A و B

$y'_p = A\cos t - B\sin t \qquad y''_p = -A\sin t - B\cos t$

$(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t$

$-5A\sin t - 5B\cos t - 3A\cos t + 3B\sin t = 2\sin t$

$\sin t(-5A + 3B) + \cos t(-5B - 3A) = 2\sin t$

$(-5A + 3B = 2) * 5 \Rightarrow -25A + 15B = 10 \Rightarrow A = -5/17$

$(-5B - 3A = 0) * 3 \qquad -9A - 15B = 0 \qquad B = 3/17$

$y_p = \dfrac{-5\sin t}{17} + \dfrac{3\cos t}{17}$

$y = y_h + y_p \Rightarrow \boxed{y = C_1 e^{-t} + C_2 e^{4t} - \dfrac{5\sin t + 3\cos t}{17}}$

Ex ⑥. $y'' + y = t(1+\sin t)$.

$y'' + y = t + t\sin t$, $P_h = r^2 + 1 = 0 \Rightarrow r = +i, r = -i$

$y_h = C_1 e^{0t} \cos(t) + C_2 e^{0t} \sin(t) \Rightarrow \boxed{y_h = C_1 \cos t + C_2 \sin t}$

$g_1(t) = t \Rightarrow y_1 = At + B$

$g_2(t) = t\sin t \Rightarrow y_2 = [(Ct + D)\cos t + (Et + F)\sin t] * t^1$

why $*t$?

Because $D\cos t$ and $F\sin t$ exist on y_h, so i have to multiply by t to make them independent

...

$\Rightarrow y_p = y_1 + y_2 \Rightarrow y_p = At + B + (Ct^2 + Dt)\cos t + (Et^2 + Ft)\sin$

$\boxed{4.2}$ Same as 3.5 but for higher order

① Find y_h by : 1- Char. equation

2- نوجد العوامل الأوليه لأي حد مشتقة

3- نعوض حتى تحقق المعادله $= 0$

4- نعمل قسمة طويلة او قسمة تركيبية

② Find y_p like 3.5

- القسمة التركيبية , $Ay^n + By^{n-1} + Cy^{n-2} - - - - = 0$

	r^n	r^{n-1}	r^{n-2}	- - - -
	A	B	C	- - - -
zero of equation		$A * Z$		
	A	$(B + AZ)$		

$\times$ $\times$ $\times z$

Exp :- $r^3 + r^2 - r + 2 = 0$

2 factors : $1, 2, -2, 0$

$r - 2 = 0 \Longleftarrow$ -2 يلي بتحقق المعادلة هو

	r^3	r^2	r	r^0
	1	1	-1	2
-2		-2	2	-2
	1	-1	1	$\underline{\underline{0}}$ الباقي

$$r^3 + r^2 - r + 2 = (r-2)(r^2 - r + 1) = 0$$

$$r = 2 \,,\; \frac{1}{2} + \frac{\sqrt{3}}{2} i \,,\; \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

ex : $r = \overset{1}{2}, \overset{2}{2}, \overset{3}{2}, \overset{4}{3}, \overset{5,6}{2 \pm 4i}, \overset{7,8}{2 \pm 4i}, \overset{9,10}{2 \pm 4i}$

$$y = C_1 e^{2t} + C_2 t e^{2t} + C_3 t^2 e^{2t} + C_4 e^{3t}$$
$$+ C_5 e^{2t} \cos(4t) + C_6 e^{2t} \sin(4t)$$
$$+ C_7 e^{2t} t \cos(4t) + C_8 e^{2t} t \sin(4t)$$
$$+ C_9 e^{2t} t^2 \cos(4t) + C_{10} e^{2t} t^2 \cos(4t)$$

مهم :- طرق حل المعادلات ذات الدرجة العالية

1. grouping : بشوف إذا في أقواس عوامل مشتركة
2. قسمة طويلة : بقسم المعادلة على " r + the zero "
3. قسمة تركيبية : نفس فوق

$\boxed{3.6}$ Non-homogenious

general form: $y'' + P(t)y' + q(t)y = g(t)$

$y = y_h + y_p$

y_h: as in 3.5 $C_1 y_1 + C_2 y_2$

y_p: $V_1 y_1 + V_2 y_2$, where

$$V_1 = -\int \frac{y_2(t)\, g(t)}{W(y_1, y_2)} \quad , \quad V_2 = \int \frac{y_1(t)\, g(t)}{W(y_1, y_2)}$$

1- $P(t)$ and $q(t)$ are Constants :-
i find y_h and Solve normal

2- $P(t)$ and $q(t)$ are NOT Constants :-
he Should give me y or y_1 and y_2
then i find the Solutions directly by V_1 and V_2