Quiz#3 solutions.

Exercise#1 [**8 marks**].

(a) Show that the sequence $x_n = n^{-2}\sin(n^3+n+1)$ has a convergent subsequence.

Since x_n is bdd (why?!), by Bolzano Weierstrass thm, it has a convergent subsequence.

(b) Give an example of an unbounded sequence that has a convergent subsequence.

$x_n = \{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \dots$ unbdd

$x_{2n} = \frac{1}{2n} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\}$ convergent subsequence.

(c) If $x_n := \sqrt{n}$, show that $\{x_n\}$ satisfies $\lim\limits_{n\to\infty} |x_{n+1} - x_n| = 0$ but it is not a Cauchy sequence.

$\lim\limits_{n\to\infty} |x_{n+1} - x_n| = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$. However, if $m = 4n$,

then $\sqrt{4n} - \sqrt{n} = \sqrt{n}$, $\forall n \Rightarrow x_n = \sqrt{n}$ is NOT Cauchy.

(or it is Not Cauchy since it diverges).

(d) Show from the **definition** that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, then $\{|x_n - y_n|\}$ is also a Cauchy sequence.

Pf. since $\{x_n\}$ and $\{y_n\}$ are Cauchy, then $\forall \varepsilon > 0$,

$\exists N_1, N_2 \in \mathbb{N}$ s.t $m, n \geq N_1 \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}$

and $m, n \geq N_2 \Rightarrow |y_n - y_m| < \varepsilon/2$.

set $N = \max\{N_1, N_2\}$. Then if $m, n \geq N$,

we have

$$\big|\, |x_n - y_n| - |x_m - y_m| \,\big| \leq |(x_n - y_n) - (x_m - y_m)|$$
$$\leq |x_n - x_m| + |y_n - y_m|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Exercise#2 [7 marks]. Let $\{x_n\}$ be a sequence of real numbers defined by

$$x_1 = 2 \text{ and } x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n},\ n = 1, 2, \cdots.$$

(a) Show that for each $n \in \mathbb{N}$, $x_n \geq \sqrt{2}$.

0.5 • For $n=1$, $x_1 = 2 \geq \sqrt{2}$.

0.5 • spse that $x_n \geq \sqrt{2}$.

• show that $x_{n+1} \geq \sqrt{2}$. Indeed, since $x_n \geq \sqrt{2}$, then

1 $(x_n - \sqrt{2})^2 \geq 0$. This implies that $\frac{x_n^2+2}{2x_n} \geq \sqrt{2}$, i.e, $x_{n+1} \geq \sqrt{2}$. Hence, $x_n \geq \sqrt{2}$, $\forall n \in \mathbb{N}$.

(b) Show that $\{x_n\}$ is monotonically decreasing.

$$x_{n+1} - x_n = \frac{1}{2}x_n + \frac{1}{x_n} - x_n$$

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$$= \frac{2 - x_n^2}{2x_n} < 0 \text{ since } x_n \geq \sqrt{2}.$$

i.e, $x_{n+1} \leq x_n$, $\forall n$.

(c) Show that $\{x_n\}$ is convergent and compute its limit.

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$\{x_n\}$ is convergent since it is decreasing and bdd below, by MCT. let $\lim_{n\to\infty} x_n = l$. Then

$$l = \frac{1}{2}l + \frac{1}{l}$$

$$\Rightarrow l^2 = 2$$

$$\Rightarrow l = \sqrt{2} \text{ or } l = -\sqrt{2} \text{ (reject) why?}$$

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$$\therefore \lim_{n\to\infty} x_n = \sqrt{2}.$$