Ch 7. (7.1) Inverse functions and their Derivatives:

9, 37, 42

Def: A function $f(x)$ is one to one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D

or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Example:

- $f(x) = x^2$

  $f(1) = f(-1)$, but $(1) \neq (-1) \Rightarrow$ Not $(1-1)$

- $f(x) = \sin x$

  $f\left(\frac{\pi}{3}\right) = f\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, but $\frac{\pi}{3} \neq \frac{2\pi}{3}$

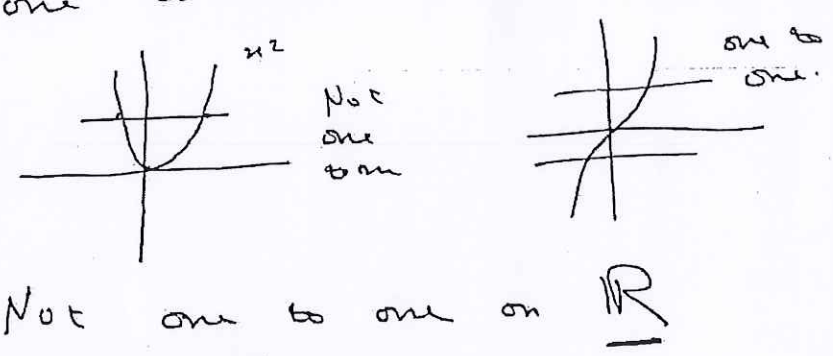
  $\Rightarrow$ $\sin x$ is Not one to one

- $f(x) = x^3$

  $f(x_1) = x_1^3 \neq f(x_2) = x_2^3 \quad \forall\, x_1 \neq x_2$

  $\Rightarrow$ x^3 is one to one

Horizontal Test



Note: that x^2 is Not one to one on $\mathbb{R}$

but If the Domain $= [0, \infty)$

$\Rightarrow$ x^2 is one to one on $[0, \infty)$

Inverse functions:

Def: Suppose that f is one to one function on a domain D with range R. The Inverse function f^{-1} is defined by $f^{-1}(b) = a$ if $f(a) = b$.

Domain (f^{-1}) = Range (f) & Range (f^{-1}) = Domain (f)

Note:
- $(f^{-1} \circ f)(x) = x$, $\forall x \in D(f)$

&
- $(f \circ f^{-1})(y) = y$, $\forall y \in D(f^{-1})$.

- $\lim\limits_{x \to c} f(x) = L$, then $\lim\limits_{x \to L} f^{-1}(x) = c$

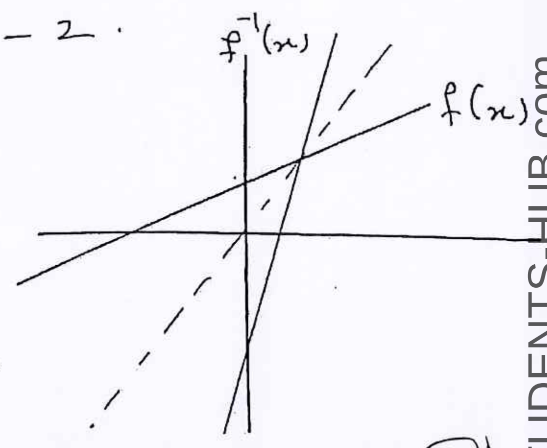
- The graphs of $f(x)$ & $f^{-1}(x)$ are symmetric about $y = x$.

Example: Find the Inverse of $y = \frac{1}{2}x + 1$

1. Solve for x in terms of y : $y = \frac{1}{2}x + 1 \Rightarrow x = 2y - 2$

2. Interchange x & y : $y = 2x - 2$.

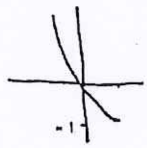
$\Rightarrow f^{-1}(x) = 2x - 2$.



Note: $f(f^{-1}(x)) = \frac{1}{2}(2x-2) + 1 = x$

& $f^{-1}(f(x)) = 2(\frac{1}{2}x + 1) - 2 = x$.

33) $f(x) = x^2 - 2x \;, \quad x \le 1$



Hint: Complete the (square)

Find f^{-1} & Identify the Range & Domain of f^{-1}

$$y = x^2 - 2x$$

$$y = f(x) = x^2 - 2x + 1 - 1$$

$$y = f(x) = (x-1)^2 - 1$$

$\rightsquigarrow$ Solve for x in terms of y

$$\rightsquigarrow \quad (x-1)^2 = y + 1$$

$$\sqrt{(x-1)^2} = \sqrt{y+1}$$

$$\rightsquigarrow \quad |x-1| = \sqrt{y+1}$$

$$\rightsquigarrow \quad 1 - x = \sqrt{y+1}$$

$$\rightsquigarrow \quad x = 1 - \sqrt{y+1}$$

$$\rightsquigarrow \quad y = 1 - \sqrt{\underbrace{x+1}_{\ge 0}} = f^{-1}(x)$$

Domain : $x \ge -1 \;, \; [-1, \infty)$

Range : $x \le 1 \;, \; (-\infty, 1]$.

Example: Find the Inverse of $y = x^2$, $x \geq 0$.

1) solve it for x $\Rightarrow y = x^2 \Rightarrow \sqrt{y} = \sqrt{x^2}$

$\Rightarrow \sqrt{y} = |x| = x$ (since $x \geq 0$)

$\Rightarrow$

$x = \sqrt{y}$

2) $\left.\begin{array}{l}\text{Interchange } x \text{ by } f^{-1}(x) \\ \& \ y \text{ by } x\end{array}\right\} \Rightarrow f^{-1}(x) = \sqrt{x}$

Derivatives of Inverse of Differentiable functions:

Thm: If f defined on I and $f'(x)$ exists and $f'(x) \neq 0$, $\forall x \in I$, then $f^{-1}(x)$ is differentiable at every point in its domain

and
$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

$$\Longleftrightarrow \quad \left.\frac{df^{-1}}{dx}\right|_{\underbrace{x=b}_{x \in D(f^{-1}) = R(f)}} = \frac{1}{\left.\frac{df}{dx}\right|_{x = f^{-1}(b)}}$$

Example: $f(x) = x^2,\ x \geq 0$ & $f^{-1}(x) = \sqrt{x}$

$f'(x) = 2x$ & $(f^{-1})'(x) = \dfrac{1}{2\sqrt{x}}$.

OR: $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{2(f^{-1}(x))} = \dfrac{1}{2\sqrt{x}}$.

If we pick $x = 2$, then. $[\ f(2) = 4\ ]$

$$(f^{-1})'(\underset{b}{4}) = \frac{1}{f'(\underbrace{f^{-1}(4)}_{f^{-1}(b)})} = \frac{1}{f'(2)} = \left.\frac{1}{2x}\right|_{x=2} = \frac{1}{4}.$$

Example: Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $\underbrace{x = 6}_{\text{يعني قيمة } y \text{ في } f(x)}$ without finding a formula of $f^{-1}(x)$.

$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{12}.$$

OR: $y = x^3 - 2 \Rightarrow x = \sqrt[3]{y+2} \Rightarrow f^{-1}(x) = \sqrt[3]{x+2}$.

$$(f^{-1})'(x) = \frac{1}{3}(x+2)^{-2/3}$$

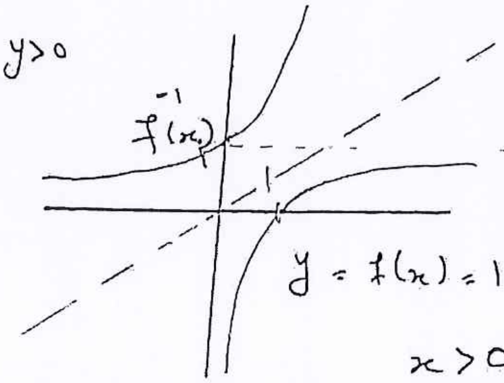
$$(f^{-1})'(6) = \frac{1}{3}(8)^{-2/3} = \frac{1}{3\sqrt[3]{(8)^2}} = \frac{1}{3(4)} = \frac{1}{12}.$$

lecture

(12) Draw f of $f(x)$, then find Domain & Range of f^{-1}

Domain f = Range f^{-1} = $\cancel{x>0} \Leftrightarrow y>0$

Range f = Domain f^{-1} = $(-\infty, 1)$



$y = f(x) = 1 - \frac{1}{x}$

$x > 0$

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(31) Find $f^{-1}(x)$ for $f(x) = \frac{x+3}{x-2}$, then find domain & Range of f^{-1} & show that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.

$y = \frac{x+3}{x-2} \Rightarrow yx - 2y = x+3 \Rightarrow yx - x = 3+2y$

$\Rightarrow x(y-1) = 3+2y \Rightarrow x = \frac{3+2y}{y-1} \Rightarrow f^{-1}(x) = \frac{3+2x}{x-1}$

Domain $f^{-1}(x) = \mathbb{R} \setminus \{1\}$ & Range $f^{-1}(x) = \mathbb{R} \setminus \{2\}$.

$$f(f^{-1}(x)) = f\left(\frac{3+2x}{x-1}\right) = \frac{\frac{3+2x}{x-1} + 3}{\frac{3+2x}{x-1} - 2} = \frac{5x}{5} = x$$

$$f^{-1}(f(x)) = \frac{3 + 2\left(\frac{x+3}{x-2}\right)}{\left(\frac{x+3}{x-2}\right) - 1} = x$$

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(41) Let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$. Find $\left.\frac{df^{-1}}{dx}\right|_{x=-1 = f(3)}$

$$\left.\frac{df^{-1}}{dx}\right|_{x=-1=f(3)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=3}} = \left.\frac{1}{3x^2 - 6x}\right|_{x=3} = \boxed{\frac{1}{9}}$$

7.2 Natural Logarithms:

Def: The natural Logarithm is the function given by:

$$\ln x = \int_1^x \frac{1}{t}\, dt \quad , \; x > 0$$

Notes: From Fundemental theorem of Calculus:

1) $\ln x$ is a Continuous function.

2) If $x > 1 \Rightarrow \ln x$ = area between $y = \frac{1}{t}$ and the x-axis from $t = 1$ to $t = x$.

3) If $0 < x < 1 \rightarrow \ln x$ = negative the area between x-axis & $y = \frac{1}{t}$ from $t = 1$ to $t = x$

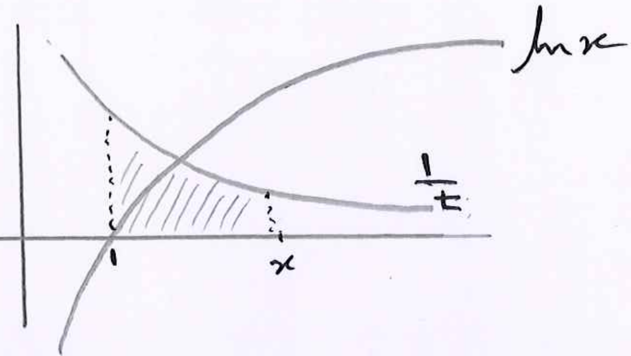
since $\ln x = \int_1^x \frac{1}{t}\, dt = -\int_x^1 \frac{1}{t}\, dt$.

4) If $x = 1$, $\ln 1 = 0$.

5) If $x \leq 0$, $\ln x$ is Undefined.

6) Domain: $(0, \infty)$.

7) Range: $\mathbb{R}$.



Def: There exists an Important number between 2 and 3 called e , such that $\ln e = 1$

(i.e) $\ln(e) = \int_1^e \frac{1}{t}\,dt = 1$

Notation: $\ln x = \log_e x$

The Derivative of $y = \ln x$.

Thm: ① $\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$. (FTC)

② $\frac{d}{dx}\ln u = \frac{1}{u}\cdot(u')$, $u > 0$.

③ $\frac{d^2}{dx^2}\ln x = \frac{-1}{x^2}$

Notes: from ① we conclude $\ln x$ is Increasing function since $x > 0 \Rightarrow \frac{1}{x} > 0$, $\forall x > 0$.

from ③ we conclude that $\ln x$ is Concave down.

Example: 1) $\frac{d}{dx} \ln 2x = \frac{1}{2x} \cdot 2 = \frac{1}{x}$, $x > 0$

2) $\frac{d}{dx} \ln(-2x) = \frac{1}{-2x} \cdot (-2) = \frac{1}{x}$, $x < 0$

3) $\frac{d}{dx} \ln \underset{x>0}{x} = \frac{d}{dx} \ln(\underset{x<0}{-x}) = \frac{1}{x}$, $x \neq 0$

4) $\frac{d}{dx} \ln(x^2+3) = \frac{1}{x^2+3}(2x) = \frac{2x}{x^2+3}$.

Properties of Logarithms:

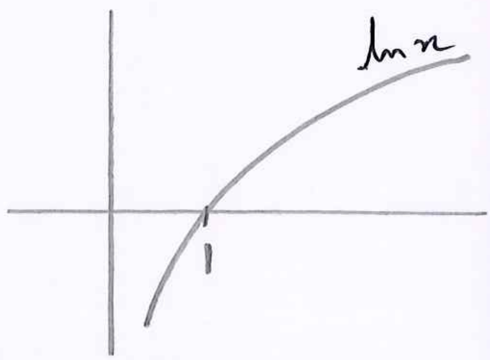
Thm: 1) $\ln(bx) = \ln b + \ln x$

2) $\ln\left(\frac{b}{x}\right) = \ln b - \ln x$

3) $\ln\left(\frac{1}{x}\right) = \ln 1 - \ln x = -\ln x$

4) $\ln(x^r) = r \ln x$.

5) $\lim\limits_{x \to 0^+} \ln x = -\infty$



6) $\lim\limits_{x \to \infty} \ln x = \infty$

7) $\int \frac{1}{u}\, du = \ln |u| + C$.

Example: ① $\ln 4 + \ln(\sin x) = \ln(4 \sin x)$

② $\ln \frac{(x+1)}{(2x-3)} = \ln(x+1) - \ln(2x-3)$.

③ $\ln \frac{1}{8} = \ln 1 - \ln 8 = -\ln 8 = -3\ln 2$.

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proof: ① Need to show $\ln bx = \ln b + \ln x$.

Note that $\underbrace{\frac{d}{dx} \ln bx}_{} = \frac{b}{bx} = \frac{1}{x} = \underbrace{\frac{d \ln x}{dx}}_{}$.

$=$

Using the following Corollary:

If $f'(x) = g'(x)$, $\forall x \in (a,b)$, then

$\exists\, C$ (s.t) $f(x) = g(x) + C$, $\forall x \in (a,b)$.

$\Rightarrow$ $\ln bx = \ln x + C$, $\forall x > 0$

Assume $x = 1$, then:

$\ln b = \ln 1 + C = 0 + C = C$

$\Rightarrow$ $\ln b = C$

$\Rightarrow$ $\ln bx = \ln x + \ln b$.

Corollary: $$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C_1 .$$

proof: Let $u = f(x) \Rightarrow du = f'(x)\,dx$

$$\Rightarrow \int \frac{du}{u} = \ln|u| + C_1 = \ln|f(x)| + C_1$$

Example: $$\int_0^2 \frac{2x}{x^2-5}\,dx = \ln|x^2-5|\Big|_0^2$$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5 .$$

Example: $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx =$$

$$= -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + C_1 .$$

$$= \ln|\cos x|^{-1} + C$$

$$= \ln|\sec x| + C$$

Example: $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$

$= \ln|\sin x| + C_1 = -\ln|\csc x| + C$

Example: $\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$

$= \int \left(\frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \right) dx$

$= \ln|\sec x + \tan x| + C.$

Example: $\int \csc x \, dx = \int \csc x \left(\frac{\csc x + \cot x}{\csc x + \cot x} \right) dx$

$= \int \left(\frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \right) dx$

$= -\ln|\csc x + \cot x| + C.$

Example: $\int_0^{\frac{\pi}{6}} \tan 2x \, dx = \int_0^{\frac{\pi}{6}} \left(\frac{-2}{-2}\right)\left(\frac{\sin 2x}{\cos 2x}\right) dx$

$= -\frac{1}{2} \ln |\cos 2x| \Big|_0^{\frac{\pi}{6}} = -\frac{1}{2}\left(\ln \frac{1}{2} - 1\right) = \frac{1}{2} \ln 2$

Example: Use logarithm differentiation to find

$\frac{dy}{dx}$ if $y = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{(x-1)}$, $x > 1$

sol: $\ln y = \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$.

$\Rightarrow \frac{1}{y} \cdot y' = \frac{2x}{(x^2+1)} + \frac{1}{2(x+3)} - \frac{1}{(x-1)}$.

$\Rightarrow y' = y\left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{(x-1)}\right)$

$\Rightarrow y' = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{(x-1)}\left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{(x-1)}\right)$.

Lecture

(23) Find the Derivative of y w.r.t x

$$y = \ln(\ln x) \Rightarrow \frac{dy}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$$

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(46) Evaluate $\displaystyle\int_{2}^{16} \frac{dx}{2x\sqrt{\ln x}}$

Let $u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow du = \dfrac{1}{x}dx$

$$\Rightarrow \int_{\ln 2}^{\ln 16} \frac{1}{2} \frac{du}{\sqrt{u}} = \int_{\ln 2}^{\ln 16} \frac{1}{2} u^{-\frac{1}{2}}\, du = \frac{1}{2} \cdot 2u^{\frac{1}{2}} \Big]_{\ln 2}^{\ln 16}$$

$$= \sqrt{\ln 16} - \sqrt{\ln 2} = \sqrt{\ln 2^4} - \sqrt{\ln 2} = 2\sqrt{\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}.$$

Use logarithim differentiation to find the derivative of y.

(64)

$$y = \frac{\theta \sin\theta}{\sqrt{\sec\theta}}$$

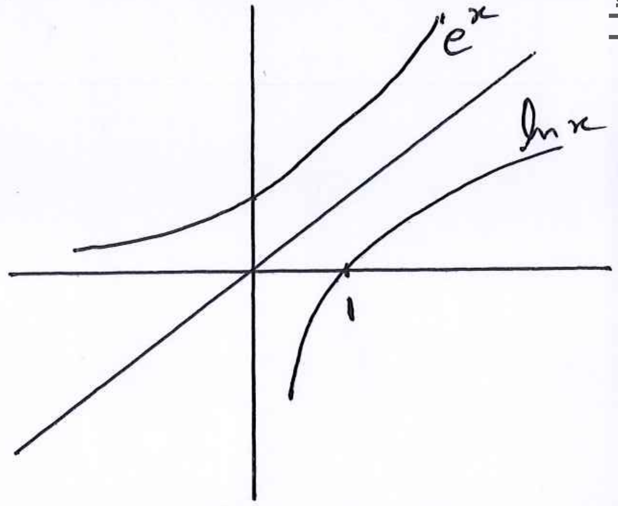
$$\ln y = \ln\theta + \ln\sin\theta - \frac{1}{2}\ln(\sec\theta).$$

$$\Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{1}{\theta} + \frac{1}{\sin\theta} \cdot \cos\theta - \frac{1}{2}\frac{1}{\sec\theta}(\sec\theta\tan\theta).$$

$$\Rightarrow \frac{dy}{d\theta} = \left[\frac{\theta\sin\theta}{\sqrt{\sec\theta}}\right]\left[\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right].$$

7.3 Exponential Functions:

Def: The Inverse of $\ln x$ is the exponential function e^x, $(\ln x)^{-1} = e^x$.



Notes:

① $\text{Domain}(e^x) = \text{Range}(\ln x) = \mathbb{R}$

② $\text{Range}(e^x) = \text{Domain}(\ln x)$
$= (0, \infty)$

③ $\lim_{x \to \infty} e^x = \infty$ & $\lim_{x \to -\infty} e^x = 0$.

④ From (3), we conclude $y = 0$ is a Horizontal Asymp. for e^x.

Def:

$\boxed{1}$ $e^{\ln x} = x$, $\forall x > 0$

$\boxed{2}$ $\ln(e^x) = x$, $\forall x$

} Inverse of each other.

$\boxed{3}$ $\ln e^r = r \ln e = r$.

Example: Solve the equation $e^{2x-6} = 4$, for x.

$$\ln\left(e^{2x-6}\right) = \ln 4$$

$$\Rightarrow \quad (2x-6)(\ln e) = \ln 4$$

$$\Rightarrow \quad 2x = \ln 4 + 6 \Rightarrow x = 3 + \tfrac{1}{2}\ln 4 = 3 + \ln 2.$$

The Derivative of e^x

$$\ln(e^x) = x$$

$$\frac{d}{dx}\ln e^x = 1 \quad \Rightarrow \quad \frac{1}{e^x}\cdot\frac{d}{dx}(e^x) = 1$$

$$\Rightarrow \quad \frac{d\,e^x}{dx} = e^x$$

There fore: $$\frac{d\,e^u}{dx} = e^u \cdot \frac{du}{dx}$$

Example ① $\frac{d}{dx}(5e^x) = 5e^x.$

② $\frac{d}{dx}e^{-x} = -e^{-x}.$

③ $\frac{d}{dx}e^{\sqrt{3x+1}} = e^{\sqrt{3x+1}}\,\frac{1\cdot 3}{2\sqrt{3x+1}}$

④ $\frac{d}{dx}e^{\sin x} = e^{\sin x}\cdot\cos x.$

The Integral of the Exponential Function:

$$\int e^u \, du = e^u + C.$$

Example: $\displaystyle\int_0^{\ln 2} e^{3x} \, dx = \frac{e^{3x}}{3}\Bigg|_0^{\ln 2} = \frac{e^{3\ln 2}}{3} - \frac{e^0}{3}$

$$= \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3}}.$$

(We do change of variables:

Let $u = 3x \Rightarrow du = 3dx$) $\Rightarrow \displaystyle\int_0^{\ln 8} e^u \cdot \frac{1}{3} \, du.$

Example: $\displaystyle\int_0^{\frac{\pi}{2}} e^{\sin x} (\cos x) \, dx$

Let $\left.\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array}\right\}$, when $x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{2} \Rightarrow u = 1$

$\Rightarrow \displaystyle\int_0^1 e^u \, du = e^u \Big|_0^1 = e - 1$

Example: $\displaystyle\int_1^4 \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$

Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$ & when $x = 1 \Rightarrow u = 1$, $x = 4 \Rightarrow u = 2$

$$\Rightarrow \int_1^2 e^u\,du = e^u \Big|_1^2 = e^2 - e$$

Thm: ① $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$.

② $e^{-x} = \frac{1}{e^x}$.

③ $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$.

④ $(e^{x_1})^r = e^{x_1 r}$.

proof: ① Let $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$, then

$\ln y_1 = x_1$ and $\ln y_2 = x_2$

$x_1 + x_2 = \ln y_1 + \ln y_2 = \ln(y_1 y_2)$

$\Longrightarrow e^{x_1 + x_2} = y_1 y_2 = e^{x_1} e^{x_2}$.

Def: For any number $a>0$ and x, the exponential function with base a is

$$a^{x} = e^{x\ln a} = e^{\ln a^{x}}.$$

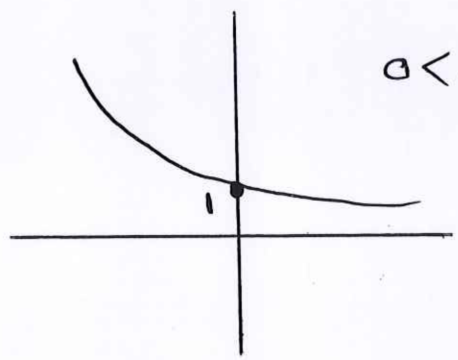
Note: The previous theorem is Valid for a^{x}.

(i-e)

$$a^{x_1} a^{x_2} = a^{x_1+x_2}.$$

$$a^{n} \cdot a^{-1} = a^{n-1}.$$

Graph of $y = a^{x}$, $a>0$



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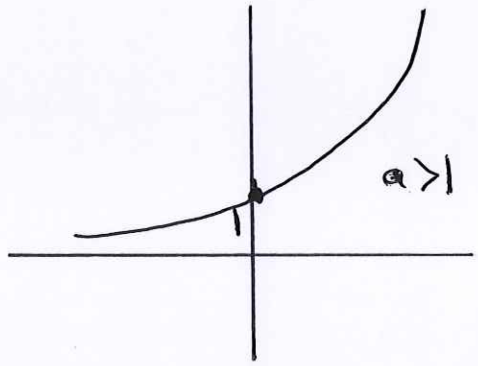

Thm: $$\frac{d\,x^{n}}{dx} = n\,x^{n-1}.$$

Since: $x^{n} = e^{n\ln x} \Longrightarrow (x^{n})' = e^{n\ln x}\cdot \frac{n}{x} = n\,\frac{x^{n}}{x}.$

Example: Find y' for $y = x^x$, $x > 0$.

Let $y = x^x = e^{x \ln x}$

$\Rightarrow y' = e^{x \ln x} \cdot \left(x\left(\frac{1}{x}\right) + \ln x \right) = x^x \left(1 + \ln x \right)$.

Thm: $e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$

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proof: Let $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$.

$\Rightarrow f'(1) = 1$. Using the definition of derivative:

$$1 = f'(1) = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$\Rightarrow 1 = \lim_{x \to 0} \frac{\ln(1+x) - \cancel{\ln(1)}^{\,0}}{x} = \lim_{x \to 0} \left(\frac{1}{x} \ln(1+x) \right).$$

$$\Rightarrow 1 = \ln \left(\lim_{x \to 0} (1+x)^{\frac{1}{x}} \right)$$, since $\ln x$ is continuous

$$\Rightarrow e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}.$$

The derivative of a^u:

Thm: ① $\frac{d}{dx} a^u = a^u \cdot \ln a \cdot \frac{du}{dx}$.

② $\int a^u \, du = \frac{a^u}{\ln a} + C$.

Examples: ① $\frac{d}{dx} 3^x = 3^x \ln 3$.

② $\frac{d}{dx}(3^{-x}) = (3^{-x})(\ln 3)(-1) = (-\ln 3)\, 3^{-x}$.

③ $\frac{d}{dx} 3^{\sin x} = 3^{\sin x} (\cos x) \cdot \ln 3$.

④ $\int 2^x \, dx = \frac{2^x}{\ln 2} + C$

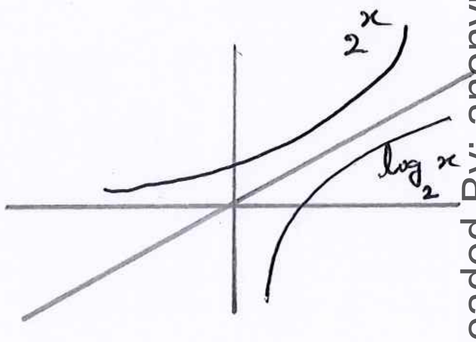
⑤ $\int 2^{\sin x} (\cos x)\, dx = \int 2^u \, du = \frac{2^u}{\ln 2} + C$

(Let $u = \sin x \Rightarrow du = \cos x \, dx$)

$= \frac{2^{\sin x}}{\ln 2} + C$

Logarithm with base a:

Def: $\log_a x = \left(a^x\right)^{-1}$



Thm: ① $a^{\log_a x} = x \quad , \quad x > 0$

② $\log_a\left(a^x\right) = x \quad , \quad \forall x .$

③ $$\log_a x = \frac{\ln x}{\ln a}$$

④ $$\frac{d}{dx}\left(\log_a u\right) = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx} .$$

Example: ① $$\frac{d}{dx} \log_{10}(3x+1) = \frac{1}{\ln 10} \cdot \frac{1}{(3x+1)} \cdot (3) .$$

② $$\int \frac{\log_2 x}{x} \, dx = \frac{1}{\ln 2} \int \frac{\ln x}{x} \, dx$$

Let $\left.\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array}\right\} \Rightarrow \quad = \frac{1}{\ln 2} \int u \, du = \frac{1}{\ln 2} \frac{(\ln x)^2}{2} + C .$

$\frac{u^2}{u}$

7.3 (26) Find dy/dx for $\ln xy = e^{x+y}$.

$\ln x + \ln y = e^{x} e^{y}$

$\frac{1}{x} + \frac{1}{y} y' = y' e^{x} e^{y} + e^{x} e^{y}$

$\Rightarrow \left(\frac{1}{y} - e^{x+y}\right) y' = e^{x+y} - \frac{1}{x}$.

$\Rightarrow y' = \dfrac{e^{x+y} - \frac{1}{x}}{\frac{1}{y} - e^{x+y}}$.

---

(42) Evaluate $\int \frac{e^{-1/x^2}}{x^3} dx$

Let $u = -x^{-2}$, then $du = -(-2) x^{-3} dx$

$\Rightarrow \int \frac{e^{u}}{x^3} \cdot \frac{x^3}{2} du = \frac{1}{2} \int e^{u} du = \frac{1}{2} e^{(-x^{-2})} + C$

---

(57) Find dy/ds for $y = 5^{\sqrt{s}}$.

$y' = 5^{\sqrt{s}} \ln 5 \cdot \left(\frac{1}{2\sqrt{s}}\right) = \left(\frac{\ln 5}{2\sqrt{s}}\right) \cdot 5^{\sqrt{s}}$

---

(92) Evaluate $\int \frac{x 2^{x^2}}{1 + 2^{x^2}} dx$

Let $u = 1 + 2^{x^2} \Rightarrow du = 2^{x^2} (2x) \ln 2 \, dx$

$\Rightarrow \frac{1}{2\ln 2} du = x 2^{x^2} dx$

$\int \frac{1}{2\ln 2 \, u} du = \frac{1}{2\ln 2} \ln|u| + C = \frac{1}{2\ln 2} \ln(1 + 2^{x^2}) + C$

7.4 Exponential change.

Exponential functions increase or decrease very rapidly with changes in the Independent variable. They describe growth or decay in many natural and industrial situations.

We will study two situations that undergo Exponential change.

1) The size of population

2) The amount of decaying radioactive material.

Assume we have the Initial Value problem:

$$\frac{dy}{dt} = ky \quad , \quad y = y_0 = y(t_0) \quad , \quad t_0 = 0.$$

$$\frac{1}{y}\, dy = k\, dt$$

$$\ln|y| = kt + c_1$$

$$|y| = e^{kt + c_1} = e^{c_1} e^{kt}$$

$$\Rightarrow \quad y = \pm e^{c_1} e^{kt} = A e^{kt} \quad , \quad \text{where } A = \pm e^{c_1}$$

$$\Rightarrow \quad y = A e^{kt} \quad \text{with} \quad y(0) = y_0$$

$$\Rightarrow \quad y_0 = A e^{k \cdot 0} = A \quad \Rightarrow \quad y(t) = y_0 e^{kt}$$

So we have $y = y_0 e^{kt}$, this quantity is said to undergo:

(i) exponential growth if $k > 0$.

(ii) exponential decay if $k < 0$.

k is called the rate constant of the Change.

Example: Suppose that in the ~~case~~ course of any given year the number of Cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?

Sol: We assume $y = y_0 e^{kt}$.

If we count from today, then $y_0 = 10,000$ at $t = 0$

$\Rightarrow \quad y = 10,000\, e^{kt}$.

when $t = 1$, The number of Cases will be 80% of its present value (8000), Hence: $(\,y(1) = 8000)$

$$8000 = 10,000\, e^{k} \Rightarrow e^{k} = 0.8$$

$$\Rightarrow \ln(e^{k}) = \ln(0.8) = k < 0$$

Now: Set $y = 1000$ and solve for t.

$$\Rightarrow \quad 1000 = 10,000\, e^{\ln(0.8)\,t}$$

$$\Rightarrow \quad e^{(\ln 0.8)t} = 0.1 \Rightarrow (\ln 0.8)\,t = \ln 0.1$$

$$\Rightarrow \quad t = \frac{\ln 0.1}{\ln 0.8} \approx 10.32 \text{ years.}$$

Radioactivity:

Some atoms are unstable and can emit mass or radiation. This process is called radioactive decay.

For example: radioactive Carbon-14 decays into nitrogen.

The decay of a radioactive element is described by:

$$\frac{dy}{dt} = -ky \quad , \quad k > 0 .$$

If $y(t_0) = y_0$, then

$$y = y_0 e^{-kt} \quad , \quad k > 0 .$$

- The half life of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. (which is constant value)

To find half-life time:

$$y_0 e^{-kt} = \frac{1}{2} y_0$$

$$e^{-kt} = \frac{1}{2} \Rightarrow -kt = \ln \frac{1}{2} = \ln 1 - \ln 2$$

$$\Rightarrow -kt = -\ln 2$$

$$\Rightarrow \boxed{t = \frac{\ln 2}{k}}$$

Example: Scientists who do carbon-14 dating use a figure of 5700 years for its half-life. Find the age of a sample in which 10 % of the radioactive nuclei originally present have decayed.

Sol: We use the decay equation $y = y_0 e^{-kt}$, $k>0$

we need to find the value of k and the value of t when y is $0.9\, y_0$ (90% of the radioactive nuclei are still present).

1) for k: $k = \frac{\ln 2}{\text{half life}} = \frac{\ln 2}{5700}$

2) for t: $y_0 e^{-kt} = 0.9\, y_0$

$\Rightarrow e^{-kt} = 0.9$

$\Rightarrow e^{-\left(\frac{\ln 2}{5700}\right) t} = 0.9$

$\Rightarrow -\frac{\ln 2}{5700} t = \ln 0.9$, therefore:

$$t = -\frac{5700 \ln 0.9}{\ln 2} \approx 866 \text{ years.}$$

Lecture

7.4 (30) Growth of bacteria. A colony of bacteria is grown under ideal Conditions in a Laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000 bacteria. How many bacteria were present initially?

Sol: $y = y_0 e^{kt}$, $y(3) = 10,000$
$y(5) = 40,000$

we need k & y_0.

$y(3) = 10,000 \Rightarrow 10,000 = y_0 e^{3k}$
$y(5) = 40,000 \Rightarrow 40,000 = y_0 e^{5k}$ $\Big\} \Rightarrow$

$y_0 e^{5k} = 4 y_0 e^{3k} \Rightarrow e^{5k} = 4e^{3k}$

$\Rightarrow e^{2k} = 4 \Rightarrow 2k = \ln 4 = 2\ln 2$

$\Rightarrow$ $k = \ln 2$

Now: $10,000 = y_0 \, e^{3\ln 2}$

$\Rightarrow y_0 = \dfrac{10,000}{e^{\ln 2^3}} = \dfrac{10,000}{8} = 1250$ bacteria.

7.5 Indeterminate Forms and L'Hopital's Rule :

Indeterminate Forms : $\left[\frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0, \infty - \infty, 1^{\infty}, 0^{0}\right]$

Thm : L'Hopital's Rule : Suppose that $f(a) = g(a) = 0$

and that $g'(x) \neq 0$, $x \neq a$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} .$$

Examples : ① $\lim_{x \to 0} \frac{3x - \sin x}{x} = 3 - \overbrace{\lim_{x \to 0} \frac{\sin x}{x}}^{=1} = \boxed{2}$

or : $\lim_{x \to 0} \frac{3x - \sin x}{x} \overset{\frac{0}{0}}{=} \lim_{x \to 0} \frac{3 - \cos x}{1} = \boxed{2}$

---

② $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} \overset{\frac{0}{0}}{=} \lim_{x \to 0} \frac{\frac{1}{2\sqrt{x+1}}}{1} = \boxed{\frac{1}{2}}$.

---

③ $\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2} \overset{\frac{0}{0}}{\underset{L.H}{=}} \lim_{x\to 0} \frac{\frac{1}{2\sqrt{x+1}} - \frac{1}{2}}{2x}$

$\overset{\frac{0}{0}}{\underset{L.H}{=}} \lim_{x\to 0} \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)(x+1)^{-\frac{3}{2}}}{2} = \lim_{x\to 0} -\frac{1}{8}\cdot\frac{1}{\sqrt[2]{(x+1)^3}} = \boxed{-\frac{1}{8}}$

Q16/

④ $\lim_{x\to 0} \frac{x - \sin x}{x^3} \overset{\frac{0}{0}}{\underset{L.H}{=}} \lim_{x\to 0} \frac{1 - \cos x}{3x^2}$

$\underset{L.H}{=} \lim_{x\to 0} \frac{\sin x}{6x} \underset{L.H}{=} \lim_{x\to 0} \frac{\cos x}{6} = \boxed{\frac{1}{6}}$.

⑤ $\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} \underset{L.H}{=} \lim_{x\to 0} \frac{\sin x}{1+2x} = \frac{0}{1} = \boxed{0}$

⑥ $\lim_{x\to 0^+} \frac{\sin x}{x^2} \underset{L.H}{=} \lim_{x\to 0^+} \frac{\cos x}{2x} = +\infty$

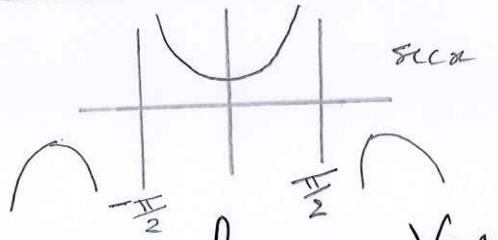
⑦ $\lim_{x\to 0^-} \frac{\sin x}{x^2} \underset{L.H}{=} \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty$

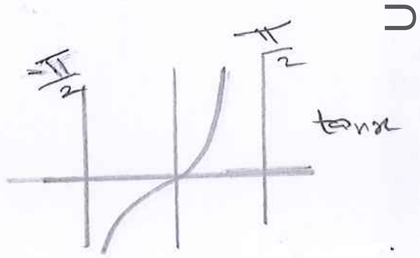
Indeterminate Forms: $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$:

Thm: If $f(x) \longrightarrow \pm\infty$, $g(x) \longrightarrow \pm\infty$ as $x \to a$

then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

$\boxed{1}$ $$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x}$$





Note that $\tan x$ & $\sec x$ have a Vertical Asymp. at $x = \frac{\pi}{2}$.

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{1+\tan x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{\cancel{\sec x}\tan x}{\sec^{\cancel{2}} x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{\sin x}{\cancel{\cos x}}}{\frac{1}{\cancel{\cos x}}}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \sin x = \boxed{1}$$

Similarly: $$\lim_{x\to\frac{\pi}{2}^+} \frac{\sec x}{1+\tan x} \overset{\frac{-\infty}{-\infty}}{=} \boxed{1}$$

$\boxed{2}$ $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} \overset{\frac{\infty}{\infty}}{\underset{\text{L.H}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{\sqrt{x}}{x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}} = \boxed{0}.$$

3 $\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\frac{\infty}{\infty}}{\underset{L.H}{=}} \lim_{x\to\infty} \frac{e^x}{2x} \overset{L.H}{=} \lim_{x\to\infty} \frac{e^x}{2} = \boxed{\infty}$

---

4 $\lim_{x\to\infty} x \sin\frac{1}{x} \overset{\infty \cdot 0}{=} \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}$

Let $\left(h = \frac{1}{x}\right) \rightarrow$ $= \lim_{h\to 0} \frac{\sin h}{h} = \boxed{1}$.

---

5 $\lim_{x\to 0^+} \sqrt{x} \ln x \overset{0.(-\infty)}{=} \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}}$

$\overset{L.H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^{3/2}}} = \lim_{x\to 0^+} \left(-2\sqrt{x}\right) = \boxed{0}$.

---

6 $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) \overset{\infty-\infty}{=} \lim_{x\to 0} \left(\frac{x - \sin x}{x \sin x}\right)$

$\overset{\frac{0}{0}}{\underset{L.H}{=}} \lim_{x\to 0} \left(\frac{1 - \cos x}{x\cos x + \sin x}\right) \overset{\frac{0}{0}}{\underset{L.H}{=}} \lim_{x\to 0} \left(\frac{\sin x}{-x\sin x + \cos x + \cos x}\right)$

$= \frac{0}{2} = \boxed{0}$.

Indeterminate Powers:

If $\lim_{x \to a} \ln f(x) = L$, then:

since e^x is continuous.

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^{\lim_{x \to a} \ln f(x)} = e^L .$$

Example: Show that $e = \lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = 1^\infty$

Let $f(x) = (1+x)^{\frac{1}{x}}$, then:

$$\ln f(x) = \frac{1}{x} \ln(1+x).$$

Now: $\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln(1+x)}{x} = \lim_{x \to 0^+} \frac{\frac{1}{x+1}}{1} = \boxed{1}.$

Now $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln f(x)} = e^{\lim_{x \to 0^+} \ln f(x)}$

$$= \boxed{e^1}$$

Example: Find $\lim_{x\to\infty} x^{\frac{1}{x}}$ (∞^0).

Let $f(x) = x^{\frac{1}{x}}$

Then $\ln f(x) = \frac{\ln x}{x}$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} \overset{L.H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Therefore: $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln x^{\frac{1}{x}}} = e^{\lim_{x\to\infty} \ln x^{\frac{1}{x}}} = e^0 = \boxed{1}$.

Q56) Find $\lim_{x\to\infty} x^{\frac{1}{\ln x}}$ (∞^0).

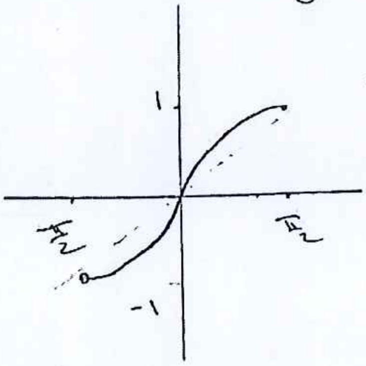
$$\lim_{x\to\infty} x^{\frac{1}{\ln x}} = \lim_{x\to\infty} e^{\ln (x)^{\frac{1}{\ln x}}} = \lim_{x\to\infty} e^{\frac{\ln x}{\ln x}} = \boxed{e^1}.$$

Q68) $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \lim_{x\to 0^+} \sqrt{\frac{1}{\frac{\sin x}{x}}} = \sqrt{\frac{1}{\lim_{x\to 0^+} \frac{\sin x}{x}}}$

$= \sqrt{\frac{1}{1}} = \boxed{1}$.

7.6. Inverse trigonometric functions: 2018/2019

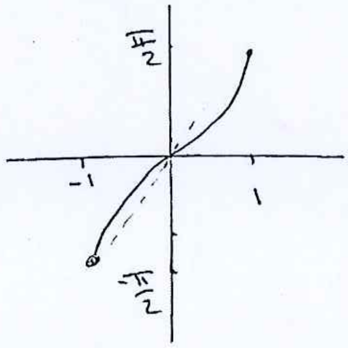
The six trigonometric functions are not one to one, However, we can restrict their domains to Intervals on which they are one to one, and then they have Inverse.



$y = \sin x$

$D = [-\frac{\pi}{2}, \frac{\pi}{2}]$

$R = [-1, 1]$

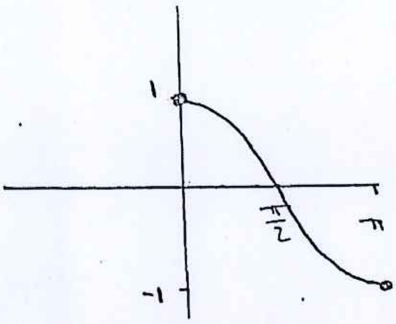


$y = \sin^{-1} x$

$D = [-1, 1]$

$R = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

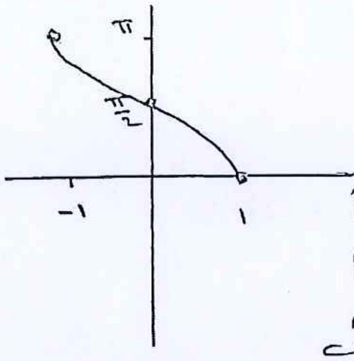
بالأول والرابع!



$y = \cos x$

$D = [0, \pi]$

$R = [-1, 1]$

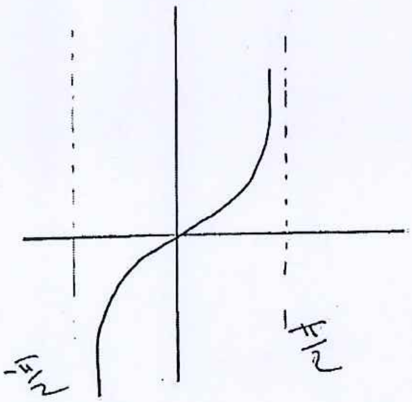


$y = \cos^{-1} x$

$D = [-1, 1]$

$R = [0, \pi]$

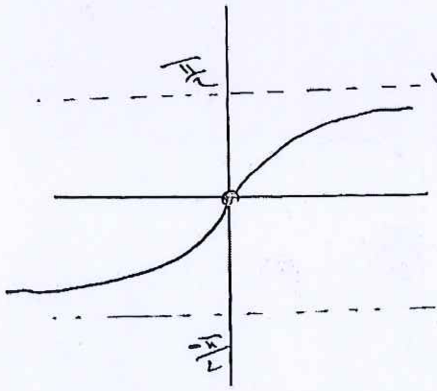
بالأول والثاني!



$y = \tan x$

$D = (-\frac{\pi}{2}, \frac{\pi}{2})$

$R = (-\infty, \infty)$



$y = \tan^{-1} x$

$D = (-\infty, \infty)$

$R = (-\frac{\pi}{2}, \frac{\pi}{2})$

بالأول والرابع!

Note that the vertical Asymptotes for $\tan x$, become Horizontal Asymptotes for $\tan^{-1} x$.

Besides: • $\tan^{-1} 0 = 0$ & $\tan 0 = 0$

• $\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty \Rightarrow \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}^-$

• $\lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty \Rightarrow \lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}^+$

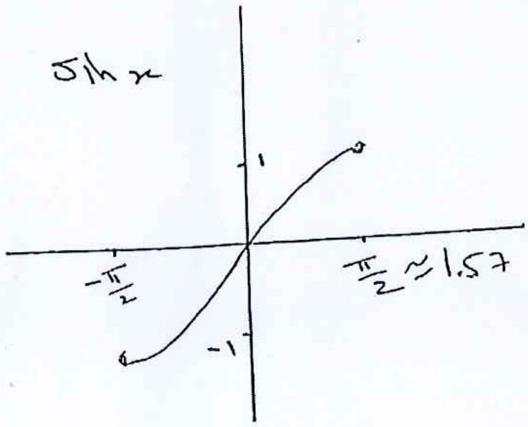
نقط على الـ [illegible]

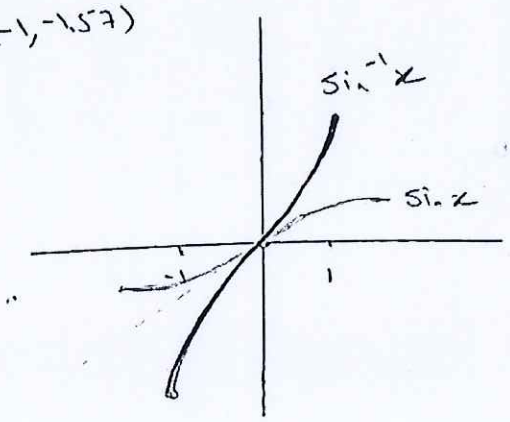
How to graph the Inverse of trigonometric functions:

$(\frac{\pi}{2}, 1) \to (1, 1.57)$

$(-\frac{\pi}{2}, -1) \to (-1, -1.57)$

1) $\sin x$



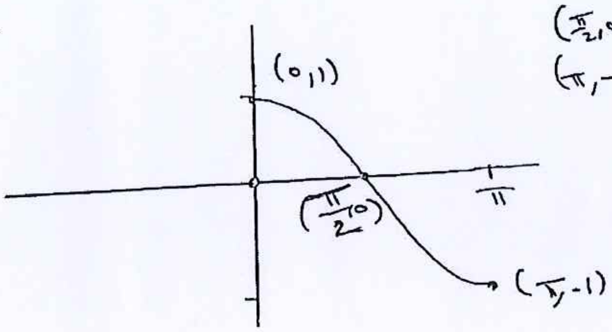


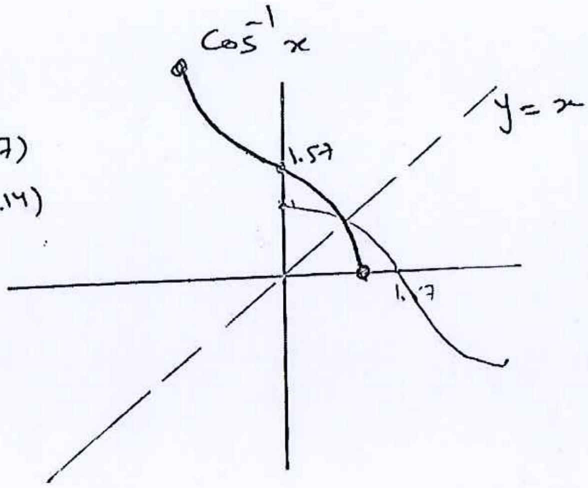
2) $\cos x$

$(0,1) \to (1,0)$

$(\frac{\pi}{2}, 0) \to (0, 1.57)$

$(\pi, -1) \to (-1, 3.14)$

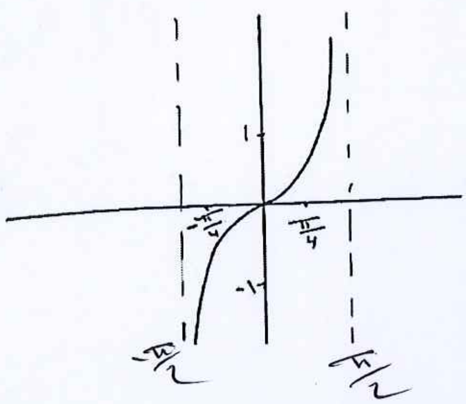


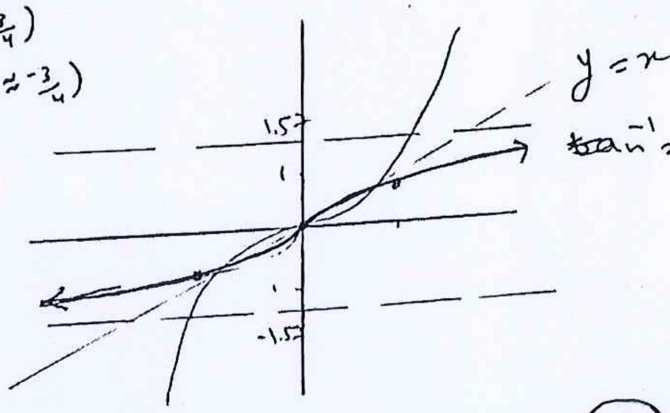


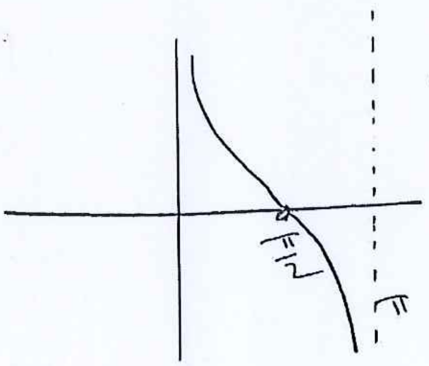
3) $\tan x$

$(\frac{\pi}{4}, 1) \to (1, \approx\frac{3}{4})$

$(-\frac{\pi}{4}, -1) \to (-1, \approx -\frac{3}{4})$



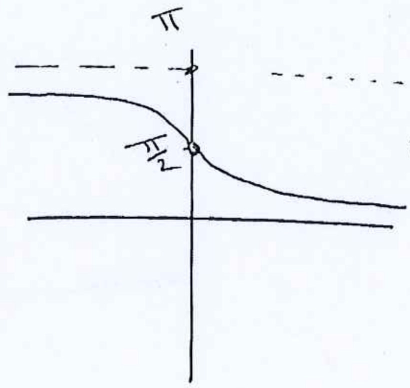




$y = \cot x$

$D = (0, \pi)$

$R = (-\infty, \infty)$



$\cot^{-1} x$

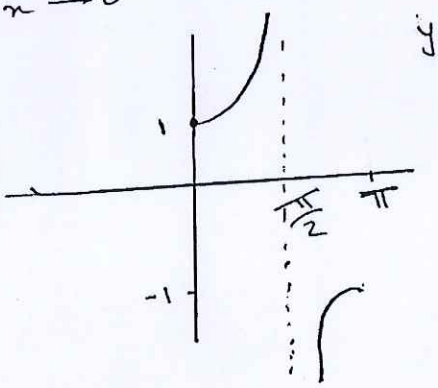
$D = (-\infty, \infty)$

$R = (0, \pi)$.

Note that $x = \pi$ is a vertical Asymptote for $\cot x$

$\Rightarrow$ $y = \pi$ is a Horizontal Asymptote for $\cot^{-1} x$

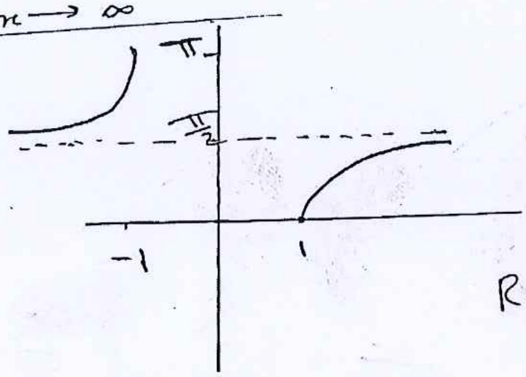
- $\cot \frac{\pi}{2} = 0 \rightsquigarrow \cot^{-1} 0 = \frac{\pi}{2}$
- $\lim\limits_{x \to \pi^-} \cot x = -\infty \rightsquigarrow \lim\limits_{x \to -\infty} \cot^{-1} x = \pi$
- $\lim\limits_{x \to 0} \cot x = \infty \rightsquigarrow \lim\limits_{x \to \infty} \cot^{-1} x = 0$



$y = \sec x$

$D = [0, \pi] \setminus \{\frac{\pi}{2}\}$

$R = (-\infty, -1] \cup [1, \infty)$

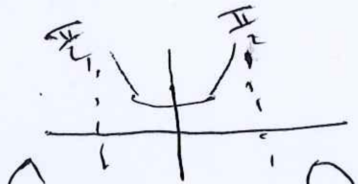


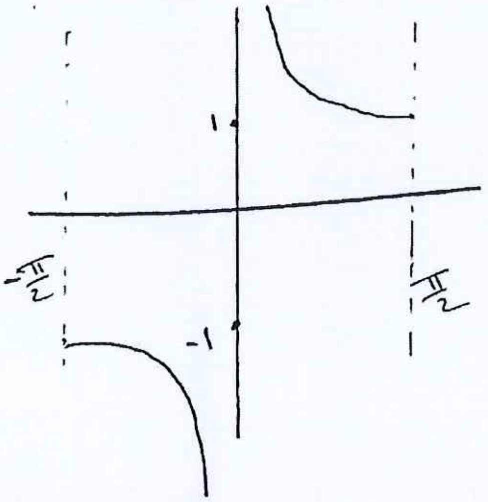
$\sec^{-1} x$

$D = (-\infty, -1] \cup [1, \infty)$

$R = [0, \pi] \setminus \{\frac{\pi}{2}\}$

- $\sec 0 = 1$, $\sec^{-1} 1 = 0$
- $\lim\limits_{x \to \frac{\pi}{2}^-} \sec x = \infty \rightsquigarrow \lim\limits_{x \to \infty} \sec^{-1} x = \frac{\pi}{2}^-$
- $\lim\limits_{x \to \frac{\pi}{2}^+} \sec x = -\infty \rightsquigarrow \lim\limits_{x \to -\infty} \sec^{-1} x = \frac{\pi}{2}^+$

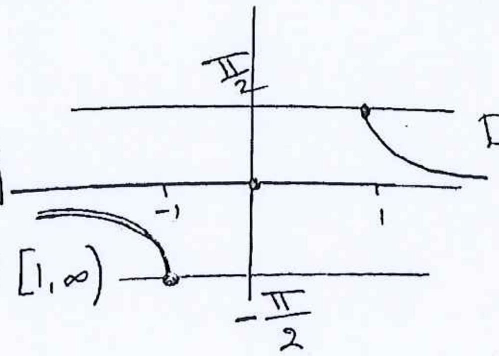




$y = \csc x$

$D = [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

$R = (-\infty, -1] \cup [1, \infty)$



$\csc^{-1} x$

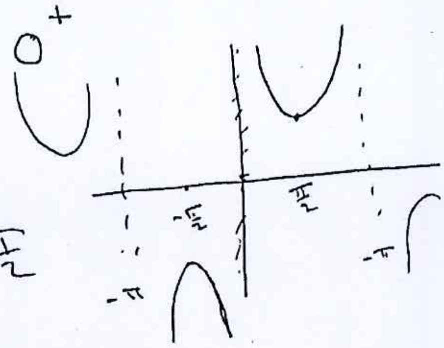
$D = (-\infty, -1] \cup [1, \infty)$

$R = [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

Zero is a vertical Asymptote for $\csc x$

$\Rightarrow$ Zero is a horizontal Asymptote for $\csc^{-1} x$

$\csc x$



- $\lim_{x \to 0^+} \csc x = \infty \Rightarrow \lim_{x \to \infty} \csc^{-1} x = 0^+$

- $\csc \frac{\pi}{2} = 1 \Rightarrow \csc^{-1} 1 = \frac{\pi}{2}$

- & $\csc -\frac{\pi}{2} = -1 \Rightarrow \csc^{-1}(-1) = -\frac{\pi}{2}$

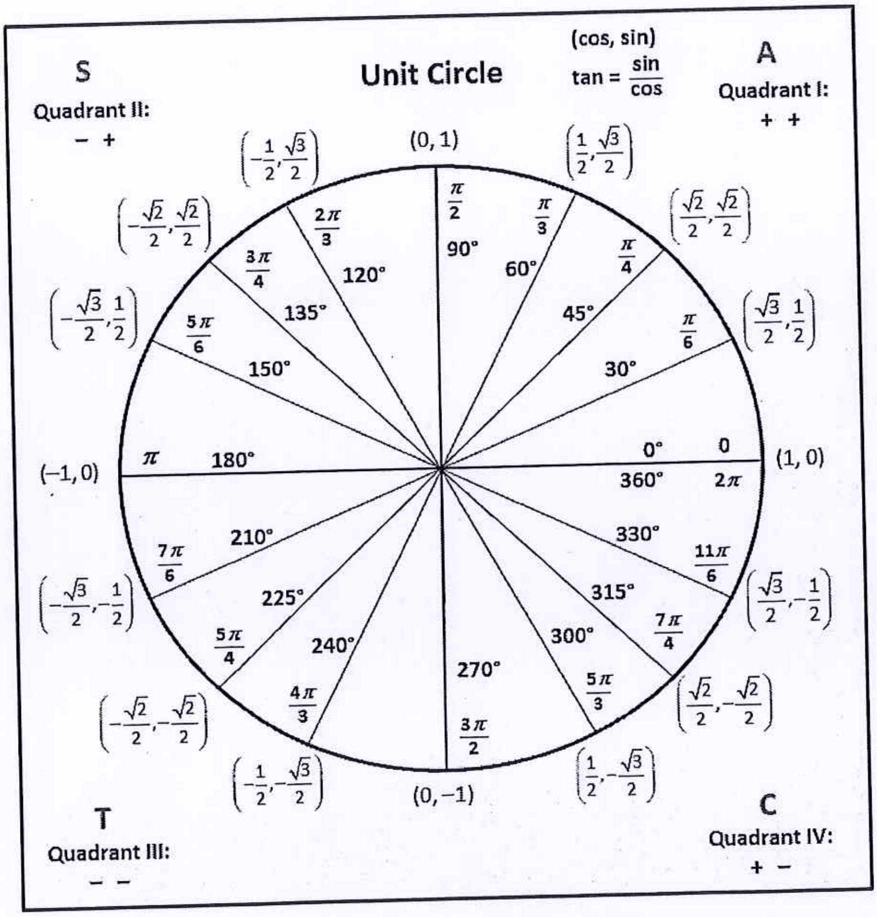
- $\lim_{x \to 0^-} \csc x = -\infty \Rightarrow \lim_{x \to -\infty} \csc^{-1} x = 0^-$

__Example:__ Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \Rightarrow \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

(2) $\cos^{-1}\left(-\frac{1}{2}\right)$ في الربع الأول والثاني فقط

$\Rightarrow \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \Rightarrow \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \in [0, \pi]$.


Unit Circle
(cos, sin)
tan = $\frac{\sin}{\cos}$
S
Quadrant II:
– +
A
Quadrant I:
+ +
T
Quadrant III:
– –
C
Quadrant IV:
+ –
(0, 1)
(–1, 0)
(1, 0)
(0, –1)
$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$
$\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$
$\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$
$\frac{\pi}{2}$ 90°
$\frac{\pi}{3}$ 60°
$\frac{\pi}{4}$ 45°
$\frac{\pi}{6}$ 30°
0° 0
360° 2π
$\frac{2\pi}{3}$ 120°
$\frac{3\pi}{4}$ 135°
$\frac{5\pi}{6}$ 150°
π 180°
$\frac{7\pi}{6}$ 210°
$\frac{5\pi}{4}$ 225°
$\frac{4\pi}{3}$ 240°
270° $\frac{3\pi}{2}$
$\frac{5\pi}{3}$ 300°
$\frac{7\pi}{4}$ 315°
$\frac{11\pi}{6}$ 330°

140

x	$\sin^{-1} x$	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/3$ 60	$\pi/6$ 30
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3 = 120$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4 = 135$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6 = 150$

$\sin^{-1} x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos^{-1} x \in [0, \pi]$

———/———

$\tan^{-1} x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

Important Identities:

Negative arguments:

1) $\sin^{-1}(-x) = -\sin^{-1}(x)$ odd function

2) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

3) $\tan^{-1}(-x) = -\tan^{-1}(x)$. odd

4) $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

5) $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$.

6) $\csc^{-1}(-x) = -\csc^{-1}(x)$ odd

Complementary arguments:

7) $\cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x)$

8) $\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$

9) $\csc^{-1}(x) = \frac{\pi}{2} - \sec^{-1}(x)$

Reciprocal arguments:

10) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(x) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$

11) $\sin^{-1}\left(\frac{1}{x}\right) = \csc^{-1}(x)$

12) $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}(x)$, $x > 0$

$\;^{-1}\left(\frac{1}{\ }\right) = \cancel{\cot^{-1}(x)}$ No need!!

Derivatives of the Inverse trigonometric functions

1) $$\frac{d \sin^{-1}u}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u|<1$$

2) $$\frac{d \cos^{-1}u}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u|<1$$

3) $$\frac{d \tan^{-1}u}{dx} = \frac{1}{1+u^2} \frac{du}{dx}$$

4) $$\frac{d \cot^{-1}u}{dx} = \frac{-1}{1+u^2} \frac{du}{dx}$$

5) $$\frac{d (\sec^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}, \quad |u|>1$$

6) $$\frac{d (\csc^{-1}u)}{dx} = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx}, \quad |u|>1$$

Example:
$$\frac{d\left(\sec^{-1}(5x^4)\right)}{dx} = \frac{1}{|5x^4|\sqrt{(5x^4)^2-1}} \cdot (20x^3)$$

$$= \frac{4}{x\sqrt{25x^8-1}}$$

The Derivative of $y = \sin^{-1} x$.

We find the derivative of $y = \sin^{-1} x$.

If $f(x) = \sin x$, then $f^{-1}(x) = \sin^{-1} x$.

Then $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{\left.\dfrac{df}{dx}\right|_{x = f^{-1}(x)}}$

$= \dfrac{1}{\cos(\sin^{-1} x)}$. Note: $\cos^2 x + \sin^2 x = 1$
$\Rightarrow \cos x = \sqrt{1 - \sin^2 x}$

$= \dfrac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}}$

$= \dfrac{1}{\sqrt{1 - \underbrace{\sin(\sin^{-1} x)}_{x}\,(\underbrace{\sin(\sin^{-1} x)}_{x})}} = \dfrac{1}{\sqrt{1 - x^2}}$.

Therefore:

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1.$$

The Derivative of $y = \tan^{-1} u$:

We find the derivative of $y = \tan^{-1} x$.

If $f(x) = \tan x$, then $f^{-1}(x) = \tan^{-1} x$.

$$\Rightarrow \quad (f^{-1})'(x) = \frac{1}{\left.\frac{df}{dx}\right|_{f^{-1}(x)}} = \frac{1}{\sec^2(\tan^{-1} x)}$$

$$= \frac{1}{1 + \tan^2(\tan^{-1} x)}, \quad \text{where } \sec^2 u = 1 + \tan^2 u.$$

$$= \frac{1}{1 + x^2}, \quad \text{where } \tan(\tan^{-1} x) = x$$

Therefore: $\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$.

The Derivative of $y = \sec^{-1} u$. [Discontinuous at $x = \frac{\pi}{2}$].

Let $y = \sec^{-1} x \Rightarrow \sec y = x$.

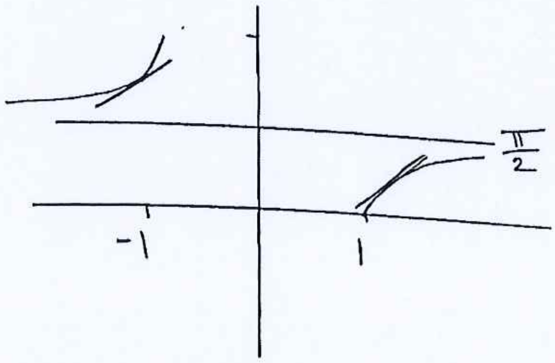
$$\Rightarrow \quad \sec y \tan y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} =$$

Now Use $\sec y = x$ and $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$

$$\Rightarrow \quad \frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}}$$

Now, from the graph of $y = \sec^{-1} x$, we note that the slope of the graph $y = \sec^{-1} x$ is always <u>positive</u>



(i-e) $$\frac{d}{dx}\left(\sec^{-1} x\right) = \begin{cases} +\dfrac{1}{x\sqrt{x^2-1}}, & x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}}, & x < -1 \end{cases}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$$

<u>Therefore:</u>

$$\frac{d}{dx}\left(\sec^{-1} u\right) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}, \quad |u| > 1.$$

Integrals of the Invers trigonometric functions:

1) $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$, $u^2 < a^2$.

2) $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$, $\forall u$

3) $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\left|\frac{u}{a}\right|\right) + C$, $|u| > a > 0$

Examples: (1) $\displaystyle\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big]_{\sqrt{2}/2}^{\sqrt{3}/2} = \overset{60}{\frac{\pi}{3}} - \overset{45}{\frac{\pi}{4}} = \frac{\pi}{12}$.

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(2) $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}} = \int \frac{dx}{\sqrt{3-(2x)^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}}$ $\left\{ \begin{array}{l} a=\sqrt{3} \\ u=2x \\ \frac{du}{2}=dx \end{array} \right.$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C = \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

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(3) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{du/u}{\sqrt{u^2-a^2}}$ $\quad u = e^x$, $du = e^x dx$, $a = \sqrt{6}$.

$$= \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

$$= \frac{1}{\sqrt{6}}\sec^{-1}\left|\frac{e^x}{\sqrt{6}}\right| + C$$

Example: Evaluate the following:

(a) $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$,

we will write $4x-x^2$ in a different formula to match any of the formulas in the table.

$$4x-x^2 = -(x^2-4x) = -(x^2-4x+4)+4$$

$$= 4-(x-2)^2 .$$

therefore, Let $a=2$, $u = x-2$ & $du = dx$.

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} = \int \frac{du}{\sqrt{a^2-u^2}}$$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C.$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C.$$

(b) $\displaystyle\int \frac{dx}{4x^2+4x+2}$

$$4x^2+4x+2 = 4(x^2+x)+2 = 4\left(x^2+x+\frac{1}{4}\right)+2-\frac{4}{4}$$

$$= 4\left(x+\frac{1}{2}\right)^2+1 = (2x+1)^2+1$$

Then:

$$\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1} =$$

Let $a=1$, $u=2x+1$, $du = 2dx$

then $$\int \frac{dx}{(2x+1)^2+1} = \frac{1}{2}\int \frac{du}{u^2+a^2} =$$

$$= \frac{1}{2}\cdot\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C_1$$

$$= \frac{1}{2}\tan^{-1}(2x+1) + C_1.$$

7.6 (12) Find $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

$$\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cot\left(-\frac{\pi}{3}\right) = \frac{-1}{\sqrt{3}}.$$

(33) Find the derivative of $y = \ln(\tan^{-1} x)$

$$\frac{dy}{dx} = \frac{1}{\tan^{-1} x} \cdot \frac{1}{1+x^2} = \frac{1}{(\tan^{-1} x)(1+x^2)}.$$

(63) Evaluate $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x \, dx}{1+e^{2x}}$.

Let $u = e^x \Rightarrow du = e^x \, dx$, $x = 0 \Rightarrow u = 1$, $x = \ln\sqrt{3}$, $u = \sqrt{3}$

$$\Rightarrow \int_1^{\sqrt{3}} \frac{u}{1+u^2} \cdot \frac{du}{u} = \int_1^{\sqrt{3}} \frac{1}{1+u^2} \, du = \tan^{-1} u \Big]_1^{\sqrt{3}} =$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

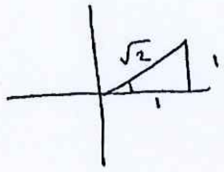
(75) Evaluate $\displaystyle\int \frac{x+4}{x^2+4} \, dx$

$$\frac{2}{2}\int \frac{x}{x^2+4} \, dx + \int \frac{4}{x^2+4} \, dx = \frac{1}{2}\ln|x^2+4| + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

(150)

7.6

6 (a) $\csc^{-1}\sqrt{2} = x$

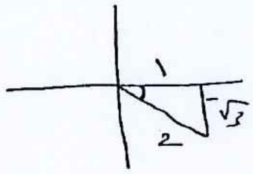


$\Leftrightarrow \quad \csc x = \sqrt{2} = \dfrac{1}{\sin x}$

$\Leftrightarrow \quad \sin x = \dfrac{1}{\sqrt{2}} \quad \Leftrightarrow$

Find x ?? $\Leftrightarrow \quad \boxed{x = \frac{\pi}{4}}$

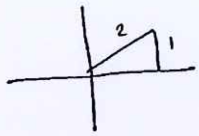
(b) $\csc^{-1}\left(\dfrac{-2}{\sqrt{3}}\right) = x$



$\Leftrightarrow \quad \csc x = \dfrac{-2}{\sqrt{3}} = \dfrac{1}{\sin x}$ اول ورابع

$\Leftrightarrow \quad \sin x = -\dfrac{\sqrt{3}}{2} \quad \Leftrightarrow \quad \boxed{x = -\frac{\pi}{3}}$

(c) $\csc^{-1} 2 = x \quad \Leftrightarrow \quad \dfrac{1}{\sin x} = 2$



$\Leftrightarrow \quad \sin x = \dfrac{1}{2} \quad \Leftrightarrow \quad \boxed{x = \frac{\pi}{6}}$

7.7 Hyperbolic Functions:

The hyperbolic function is a combination of the two exponential function e^x and e^{-x}. (They are numbers)

Def:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{odd}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}. \quad \text{even}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{odd}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad \text{odd.}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}. \quad \text{even}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \quad \text{odd.}$$

Note:

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = -\sinh(x) \quad \text{(odd)}$$

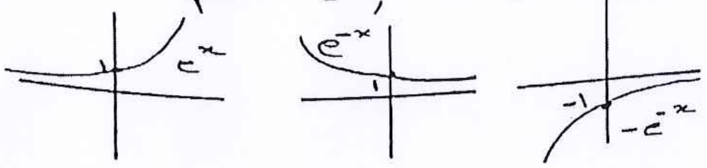
$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh(x) \quad \text{(even)}$$

The six basic hyperbolic functions

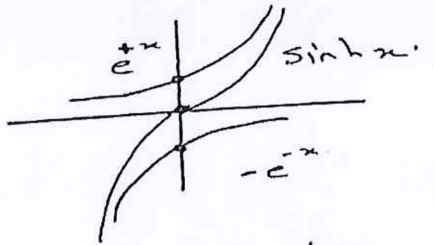
How to draw the hyperbolic functions?

1) Take $\sinh x = \frac{e^{x} - e^{-x}}{2}$

we plot e^{x}, e^{-x}, $-e^{-x}$



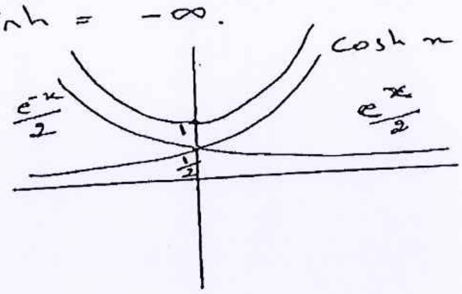
$\Rightarrow$



Since, we divide $\frac{e^{x}}{2}$, e^{-x} over 2 also, they got closer more over $\sinh 0 = 0$

Note: $\sinh x$ is an odd function

& $\lim_{x \to \infty} \sinh x = \infty$ & $\lim_{x \to -\infty} \sinh = -\infty$.



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2) Now: $\cosh x = \frac{e^{x} + e^{-x}}{2}$

$\lim_{x \to \pm\infty} \cosh x = \infty$

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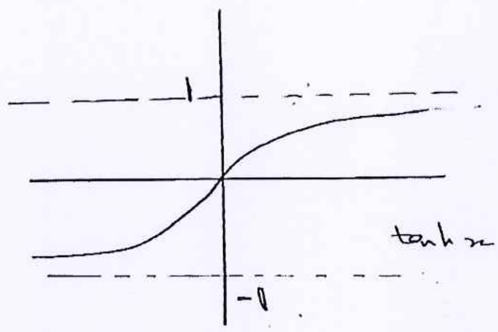
3) Now: $\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$



- for $x > 0$, as $x \to \infty$, we note that

$e^{-x} \to 0$ & $\frac{e^{x}}{e^{x}} \to 1 \Rightarrow \tanh x \to 1$

- for $x < 0$, as $x \to -\infty$,

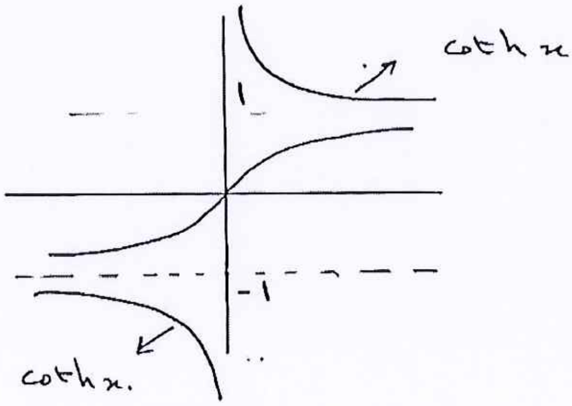
$e^{x} \to 0$ & $\tanh x \to -1$

- & $\tanh 0 = 0$ $\Rightarrow$ $y = 1$ & $y = -1$ are horizontal asymptotes

Vertical Asymp. $\lim_{x \to a^{+}} f(x) = \pm\infty$

Horizontal Asym $\lim_{x \to \pm\infty} f(x) = L$

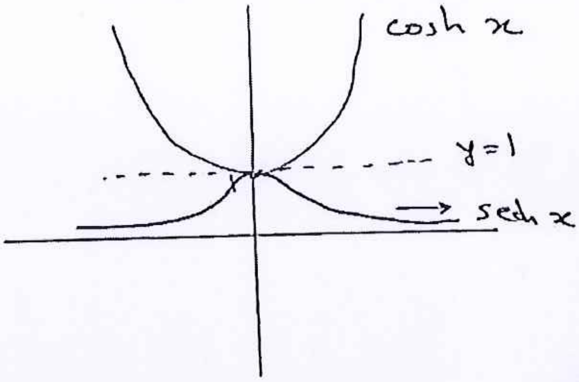
4) $\underline{\coth x} = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}} = \dfrac{1}{\tanh x}$



- $\lim_{x \to 0^+} \coth x = +\infty$
- $\lim_{x \to 0^-} \coth x = -\infty$
- $\lim_{x \to \infty} \coth x = 1^+$
- $\lim_{x \to -\infty} \coth x = 1^-$

& $x = 0$ vertical Asymptote
$y = \pm 1$ are Horizontal Asymptote.

5) $\underline{\operatorname{sech} x} = \dfrac{1}{\cosh x} = \dfrac{2}{e^{x} + e^{-x}}$

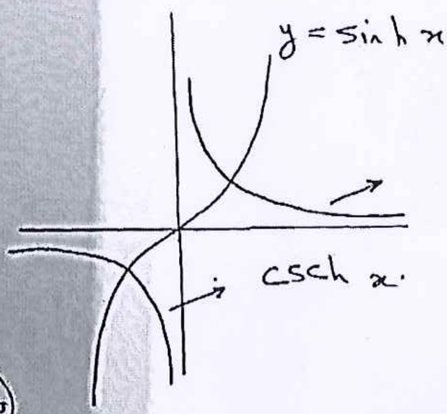


when $x = 0 \Rightarrow \operatorname{sech} 0 = 1$

$\left.\begin{array}{l} \lim_{x \to \infty} \operatorname{sech} x = 0^+ \\ \lim_{x \to -\infty} \operatorname{sech} x = 0^+ \end{array}\right\}$ symmetric about y-axis

& the x-axis is horizontal Asymptote.
$y = 0$

6) $\underline{\operatorname{csch} x} = \dfrac{1}{\sinh x} = \dfrac{2}{e^{x} - e^{-x}}$



- when $x = 0$, we have undefined value
- Therefor we will study the Behavior.
- $\lim_{x \to 0^+} \operatorname{csch} x = +\infty$
- $\lim_{x \to 0^-} \operatorname{csch} x = -\infty$
- $\lim_{x \to \infty} \operatorname{csch} x = 0^+$

& $x = 0$, $y = 0$ are Asymptotes

· Identities for hyperbolic functions:

- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh 2x = 2\sinh x \cosh x$
- $\cosh^2 x = \dfrac{\cosh(2x)+1}{2}$
- $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
- $\tanh^2 x = 1 - \operatorname{sech}^2 x$
- $\coth^2 x = 1 + \operatorname{csch}^2 x.$

proof by definitions

Derivatives of hyperbolic functions:

- $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$
- $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$
- $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$
- $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$
- $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$
- $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

proof: $\frac{d}{dx}(\operatorname{csch} u) = \frac{d}{dx}\left(\frac{1}{\sinh u}\right) = -\frac{\cosh u}{\sinh^2 u} \cdot \frac{du}{dx}$

$= \frac{-1}{\sinh u} \cdot \frac{\cosh u}{\sinh u} \cdot \frac{du}{dx} = -\operatorname{csch} u \coth u \cdot \frac{du}{dx}$.

Integral formulas for hyperbolic functions:

1) $\int \sinh u \, du = \cosh u + C$

2) $\int \cosh u \, du = \sinh u + C$

3) $\int \operatorname{sech}^2 u \, du = \tanh u + C$

4) $\int \operatorname{csch}^2 u \, du = -\coth u + C$

5) $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$

6) $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$

Q 15) $y = 2\sqrt{t} \tanh \sqrt{t}$

$y' = (2\sqrt{t}) \operatorname{sech}^2 t \left(\frac{1}{2\sqrt{t}}\right) + \frac{2 \tanh\sqrt{t}}{2\sqrt{t}} =$

$= \operatorname{sech}^2 t + \frac{\tanh \sqrt{t}}{\sqrt{t}}$.

Example: $\frac{d}{dt}\operatorname{sech}\left(\sqrt{1+t^2}\right) =$

$$= -\operatorname{sech}\left(\sqrt{1+t^2}\right)\tanh\left(\sqrt{1+t^2}\right)\cdot\left(\frac{2t}{2\sqrt{1+t^2}}\right).$$

Example: $\int \coth 5x \, dx = \frac{1}{5}\int \frac{5\cosh 5x}{\sinh 5x}\,dx$

$$= \frac{1}{5}\ln|\sinh 5x| + C.$$

Example: $\int_0^1 \sinh^2 x \, dx = \int_0^1 \frac{\cosh 2x - 1}{2}\,dx$

$$= \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1 = \frac{1}{2}\left[\frac{\sinh 2}{2} - 1\right].$$

Q54) $\int_0^{\ln 2} 4e^{-\theta}\sinh\theta \, d\theta = \int_0^{\ln 2} 4e^{-\theta}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)d\theta$

$$= \int_0^{\ln 2}\left(2 - 2e^{-2\theta}\right)d\theta = 2\theta - \frac{2e^{-2\theta}}{-2}\Big]_0^{\ln 2}$$

$$= 2\theta + e^{-2\theta}\Big|_0^{\ln 2} = 2\ln 2 + e^{-2\ln 2} - 1$$

$$= \ln 4 + \frac{1}{4} - 1 = \boxed{\ln 4 - \frac{3}{4}}$$

7.8 Relative Rates of Growth:

Rates of Growth as $x \longrightarrow \infty$

Def: Let $f(x)$ and $g(x)$ be positive for x sufficiently large:

1) f grows faster than g as $x \longrightarrow \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty \iff \lim_{x \to \infty} \frac{g(x)}{f(x)} = 0$$

2) f and g grow at the same rate as $x \longrightarrow \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$$, where L is finite & positive.

Note: $f(x) = 2x$ & $g(x) = x$ grow at the same rate.

properties: $(x \to \infty)$:

(1) If $a > 1$, then a^x is faster than x^r, $\forall r > 0$

Example: e^x grows faster than x^2 as $x \longrightarrow \infty$

$$\lim_{x \to \infty} \frac{e^x}{x^2} \overset{L.H}{=} \lim_{x \to \infty} \frac{e^x}{2x} \overset{L.H}{=} \lim_{x \to \infty} \frac{e^x}{2} = \infty$$

(2) If $a > b > 0$, then a^x is faster than b^x:

Example: 3^x grows faster than 2^x as $x \to \infty$.

$$\lim_{x \to \infty} \frac{3^x}{2^x} = \lim_{x \to \infty} \left(\frac{3}{2}\right)^x = \infty$$

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(3) x^r is faster than $\log_a(x)$ & $\ln x$, $\underline{\forall r} > 0$

Example: x^2 grows faster than $\ln x$ as $x \to \infty$

$$\lim_{x \to \infty} \frac{x^2}{\ln x} \overset{L.H}{=} \lim_{x \to \infty} \frac{2x}{1/x} = \lim_{x \to \infty} 2x^2 = \infty$$

Example: $\ln x$ grows slower than $x^{\frac{1}{n}}$ as $x \to \infty$

$$\lim_{n \to \infty} \frac{\ln x}{x^{\frac{1}{n}}} \overset{L.H}{=} \lim_{n \to \infty} \frac{\frac{1}{x}}{\frac{1}{n} x^{\frac{1}{n}-1}} = \lim_{n \to \infty} \frac{n}{x^{\frac{1}{n}}} = 0$$

---

(4) $\log_a x$ & $\log_b x$ with a & $b > 1$

always grow at the same rate. xs $x \to \infty$

$$\lim_{x \to \infty} \frac{\log_a x}{\log_b x} = \lim_{x \to \infty} \frac{\ln x / \ln a}{\ln x / \ln b} = \frac{\ln b}{\ln a}.$$

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(5) x^x is faster than a^x, $\forall a > 0$ as $x \to \infty$

~~Example~~: $$\lim_{x \to \infty} \frac{x^x}{a^x} = \lim_{x \to \infty} \left(\frac{x}{a}\right)^x = \infty \quad (\text{why})?$$

(6) If f and g have the same rate of grow and g and h have the same rate of grow then f and h grow at the same rate.

<u>Example:</u> $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L_1$ & $\lim_{x\to\infty} \frac{g(x)}{h(x)} = L_2$

then $\lim_{x\to\infty} \frac{f(x)}{h(x)} = L_1 . L_2 = L.$

<u>Example:</u> show that $\sqrt{x^2+5}$ and $(2\sqrt{x}-1)^2$ grow at the same rate as $x \to \infty$.

$\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{(2(\sqrt{x})-1)^2}$ (It's hard to compare it).

therefore: $\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{x} = \lim_{x\to\infty} \sqrt{1+\frac{5}{x^2}} = \boxed{1}$

& $\lim_{x\to\infty} \frac{(2\sqrt{x}-1)^2}{x} = \lim_{x\to\infty} \left(\frac{2\sqrt{x}-1}{\sqrt{x}}\right)^2 = \lim_{x\to\infty}\left(2-\frac{1}{\sqrt{x}}\right)^2 = \boxed{4}$

Then $\lim_{x\to\infty} \frac{\sqrt{x^2+5}\,/\,x}{(2\sqrt{x}-1)^2 / x} = \frac{1}{4}.$

OR: $\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{(2\sqrt{x}-1)^2} = \lim_{x\to\infty} \frac{x\sqrt{1+5/x^2}}{x\left(2-\frac{1}{\sqrt{x}}\right)^2} = \frac{1}{4}.$

Q8) Order the following functions from Slowest growing to fastest growing as $x \longrightarrow \infty$

a) 2^x b) x^2

c) $(\ln 2)^x$ d) e^x

Sol: $\lim\limits_{x\to\infty} \dfrac{(\ln 2)^x}{x^2} \overset{L.H}{=} \lim\limits_{x\to\infty} \dfrac{(\ln 2)^x \cdot \ln(\ln 2)}{2x}$

$\overset{L.H}{=} \lim\limits_{x\to\infty} \dfrac{(\ln 2)^x \cdot (\ln(\ln 2))^2}{2} = 0$

$\Rightarrow$ $(\ln 2)^x$ grows slower than x^2.

Now: $\lim\limits_{x\to\infty} \dfrac{x^2}{2^x} \overset{L.H}{=} \lim\limits_{x\to\infty} \dfrac{2x}{2^x(\ln 2)} \overset{L.H}{=} \lim\limits_{x\to\infty} \dfrac{2}{(2^x)(\ln 2)^2} = 0$

$\Rightarrow$ x^2 growes slower than 2^x

Now $\lim\limits_{x\to\infty} \dfrac{2^x}{e^x} = \lim\limits_{x\to\infty} \left(\dfrac{2}{e}\right)^x = 0$

$\Rightarrow$ 2^x growes slower than 2^x

Therefore : The order from slower to faster : $(\ln 2)^x, x^2, 2^x, e^x$.