7.4 Exponential Change

P.E $\qquad$ I.C

I.V.P :- $\dfrac{dy}{dt} = ky$ $\qquad$ $y = y_0 = y(t_0)$, $t_0 = 0$

$$\frac{dy}{y} = k\,dt$$

$$\ln|y| = kt + c$$

$\longrightarrow y = y_0 e^{kt}$ $\qquad$ Exponential growth , $k > 0$

$\longrightarrow y = y_0 e^{-kt}$ $\qquad$ Exponential decay , $k > 0$

Q28 Voltage in a discharging capacitor. Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage across its terminals and that if t is measured in seconds $\dfrac{dV}{dt} = \dfrac{-1}{40} V$

Sol: Find V ?? How long will it take the voltage to drop to 10% of the original value ??

$$V = V_0 e^{-\frac{1}{40}t}$$

$$10\% \cdot V_0 = V_0 e^{(-1/40)t}$$

$$0.10 = e^{(-1/40)t} \longrightarrow t = 40 \ln 10 = \ln 10^{40} \text{ sec.}$$

$\boxed{30}$ Growth of Bacteria

At the end of 3 hours there are 10000 bacteria.
At the end of 5 hours there are 40000.
How many bacteria were present initially??

<u>Sol:</u> $y = y_0 e^{kt}$ with $y(3) = 10000$ $y(5) = 40000$

$$\left.\begin{array}{l} \rightarrow 10000 = y_0 e^{3k} \\ \rightarrow 40000 = y_0 e^{5k} \end{array}\right\} \Rightarrow 4y_0 e^{3k} = y_0 e^{5k}$$

$$\Rightarrow e^{2k} = 4$$

$$\Rightarrow 2k = \ln 4 \Rightarrow \boxed{k = \ln 2}$$

$$y = y_0 e^{(\ln 2)t}$$

Now, use $y(3) = 10000$ to find y_0

$$10000 = y_0 e^{3\ln 2}$$

$$y_0 = \frac{10000}{8} = 1250 \text{ bacteria}$$

31 The incidence of a disease :— (Continuation of Exp3)

Suppose that in any given th number of cases can be reduced by 25% instead of 20 %

ⓐ How long will it take to reduce the number of cases to 1000 ??

We assume $y = y_0 e^{kt}$

If we count from today, then $y_0 = 10000$ at $t=0$

$$y = 10000\, e^{kt}$$

$$7500 = 10000\, e^{k}$$

$$0.75 = e^{k} \longrightarrow \boxed{k = \ln 0.75}$$

$k < 1$
exp. decay

Now, $y = y_0\, e^{\ln 0.75\, t}$ ----- Solve it for t.

$$1000 = 10000\, e^{\ln 0.75\, t}$$

$$\ln 0.1 = \ln 0.75\, t$$

$$\boxed{t = 8 \text{ years}}$$

ⓑ How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1 ??

$$y = 10000\, e^{\ln 0.75\, t}$$

$$1 = 10000\, e^{\ln 0.75\, t}$$

$t = 32.02$ years

time: more than 32.02 years

36 The half Life of Polonium is 139 days but your sample will not be useful to you after 95% of the sample has disintegrated. For about How many days after the sample arrives will you be able to use the polonium ??

Sol: $y = y_0 e^{-kt}$

$T = \text{half life} = 139\,\text{day} \Longrightarrow k = \frac{\ln 2}{T} = \frac{\ln 2}{139}$

$\Longrightarrow k = 0.00499$

$y = y_0 e^{-0.00499t}$

$0.05 \cancel{y_0} = \cancel{y_0} e^{-0.00499t}$

$t = \frac{\ln 0.05}{-0.0049} = 600.34$

$\boxed{t \approx 600 \text{ days}}$

$\boxed{7.5}$ Indeterminate forms and L'Hopital Rule :-

$\boxed{10}$ Use L'Hopital Rule to evalute the limit :-

$\boxed{10}$ $\lim_{t \to 1} \dfrac{t^3 - 1}{4t^3 - t - 3} = \dfrac{0}{0}$

$\lim_{t \to 1} \dfrac{3t^2}{12t^2 - 1} = \dfrac{3}{11}$

$\boxed{21}$ $\lim_{x \to 0} \dfrac{x^2}{\ln(\sec x)} = \dfrac{0}{0}$

$\lim_{x \to 0} \dfrac{2x}{\frac{1}{\sec x} \sec x \tan x} = \lim_{x \to 0} \dfrac{2x}{\tan x} = \lim_{x \to 0} \dfrac{2}{\sec^2 x}$

$= \dfrac{2}{(1)^2} = 2$

$\boxed{27}$ $\lim_{\theta \to 0} \dfrac{3^{\sin\theta} - 1}{\theta} = \dfrac{0}{0}$

$= \lim_{\theta \to 0} \dfrac{3^{\sin\theta}(\ln 3)(\cos\theta)}{1}$

$= (\ln 3)(3^0)(1) = \ln 3$

$\boxed{30}$ $\lim_{x \to 0} \frac{3^x - 1}{2^x - 1} = \frac{0}{0}$

$\lim_{x \to 0} \frac{3^x \ln 3}{2^x \ln 2} = \frac{\ln 3}{\ln 2} = \boxed{\log_2 3}$

$\boxed{34}$ $\lim_{x \to 0^+} \frac{\ln(e^x - 1)}{\ln x} = \frac{-\infty}{-\infty}$

$\lim_{x \to 0^+} \frac{e^x}{e^x - 1} \Big/ \frac{1}{x}$

$= \lim_{x \to 0^+} \frac{x e^x}{e^x - 1}$

$= \lim_{x \to 0^+} \frac{x e^x + e^x}{e^x} = \lim_{x \to 0^+} x + 1 = \boxed{1}$

$\boxed{38}$ $\lim_{x \to 0^+} \left(\ln x - \ln \sin x \right) = -\infty + \infty = \infty - \infty$

$= \lim_{x \to 0^+} \ln\left(\frac{x}{\sin x}\right)$

$= \ln \left[\lim_{x \to 0^+} \frac{x}{\sin x} \right] = \frac{0}{0}$

$= \ln \left[\lim_{x \to 0^+} \frac{1}{\cos x} \right] = \ln(1) = \boxed{0}$

Another solution:-

$\lim_{x \to 0^+} \frac{x/x}{\sin x / x}$

$= \lim_{x \to 0^+} \frac{1}{\frac{\sin x}{x}}$

$= \frac{1}{1} = \boxed{1}$

$\boxed{\ln(1) = 0}$

$\boxed{41}$ $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$ $\frac{1}{\text{small}+} - \frac{1}{\text{small}+}$

$\infty - \infty$

$$= \lim_{x\to 1^+} \frac{\ln x - x + 1}{\ln x(x-1)} \quad \frac{0}{0}$$

$$= \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{\ln x(1) + (x-1)\cdot\frac{1}{x}}$$

$$= \lim_{x\to 1^+} \frac{1-x}{x\ln x + x - 1} = \frac{0}{0}$$

$$= \lim_{x\to 1^+} \frac{-1}{\cancel{\frac{x}{x}} + \ln x + 1}$$

$$= \lim_{x\to 1^+} \frac{-1}{2+\ln x} = \frac{-1}{2+0} = \boxed{\frac{-1}{2}}$$

$\boxed{42}$ $\lim_{x\to 0^+} \left(\csc x - \cot x + \cos x\right)$ $\frac{1}{\text{small}+} - \frac{1}{\text{small}+} + 1$

$\infty - \infty + 1$

$$= \lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) + \lim_{x\to 0^+} \cos x$$

$$= \lim_{x\to 0^+} \frac{1-\cos x}{\sin x} \; \frac{0}{0} \quad + \lim_{x\to 0^+} \cos x$$

$$= \lim_{x\to 0^+} \frac{+\sin x}{\cos x} + \lim_{x\to 0^+} \cos x$$

$$= 0 + 1 = \boxed{1}$$

$\boxed{46}$ $\lim_{x \to \infty} x^2 e^{-x}$ $\quad \infty \cdot 0$

$$= \lim_{x \to \infty} \frac{x^2}{e^x} \quad \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{2x}{e^x}$$

$$= \lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = \boxed{0}$$

$\boxed{47}$ $\lim_{x \to 0} \frac{x - \sin x}{x \tan x}$ $\quad \frac{0}{0}$

$$= \lim_{x \to 0} \frac{1 - \cos x}{x \sec^2 x + \tan x} \quad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{2x \cdot \sec^2 x \tan x + \sec^2 x + \sec^2 x}$$

$$= \frac{0}{0 + 2} = \boxed{0}$$

$\boxed{52}$ $\lim_{x \to 1^+} x^{1/x-1}$ $(1)^{\infty}$

$* \quad f(x) = x^{\frac{1}{x-1}}$

$\rightarrow \ln f(x) = \frac{1}{x-1} \ln x$

$\rightarrow \lim_{x \to 1^+} \ln f(x) = \lim_{x \to 1^+} \frac{\ln x}{x-1}$ $\frac{0}{0}$

$= \lim_{x \to 1^+} \frac{\frac{1}{x}}{1}$

$= 1$

$\therefore \lim_{x \to 1^+} x^{\frac{1}{x-1}} = e^{1} = \boxed{e}$

$\boxed{53}$ $\lim_{x \to \infty} (\ln x)^{1/x}$

$* \quad f(x) = (\ln x)^{\frac{1}{x}}$

$* \quad \ln f(x) = \frac{1}{x} \ln(\ln x)$

$* \quad \lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln(\ln x)}{x}$ $\frac{\infty}{\infty}$

$= \lim_{x \to \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{1} = 0$

$\therefore \lim_{x \to \infty} f(x) = \lim_{x \to \infty} (\ln x)^{1/x} = e^{0} = 1$

$\boxed{60}$ $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x = (\infty^0)$

$* \quad f(x) = \left(1 + \frac{1}{x}\right)^x$

$* \quad \ln f(x) = x \ln\left(1 + \frac{1}{x}\right)$

$* \lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \frac{\infty}{\infty}$

$= \lim_{x \to 0^+} \frac{\frac{1}{1 + 1/x} \cdot \frac{-\cancel{1}}{\cancel{x^2}}}{\frac{-\cancel{1}}{\cancel{x^2}}}$

$= \lim_{x \to 0^+} \frac{x}{x+1}$

$= 0$

$\therefore \lim_{x \to 0^+} f(x) = \lim_{\boxed{x \to 0^+}} \left(1 + \frac{1}{x}\right)^x = e^0 = 1$

Theorem $\lim_{\boxed{x \to \infty}} \left(1 + \frac{a}{x}\right)^x = e^a$

$\boxed{62}$ $\lim_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{1/x}$ ∞^0

$* \; f(x) = \left(\frac{x^2+1}{x+2}\right)^{1/x}$

$* \; \ln f(x) = \frac{1}{x} \ln\left(\frac{x^2+1}{x+2}\right)$

$* \; \lim_{x\to\infty}(\ln f(x)) = \lim_{x\to\infty} \frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x}$

$= \lim_{x\to\infty} \frac{\ln(x^2+1) - \ln(x+2)}{x}$ $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{\frac{2x}{x^2+1} - \frac{1}{x+2}}{1}$

$= \lim_{x\to\infty} \frac{2x(x+2) - (x^2+1)}{(x^2+1)(x+2)}$ $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{2x^2+4x-x^2-1}{x^3+2x^2+x+2}$

$= \lim_{x\to\infty} \frac{4x+4-2x}{3x^2+4x+1} = \lim_{x\to\infty} \frac{2x+4}{3x^2+4x+1}$

$= \lim_{x\to\infty} \frac{2}{6x+4} = 0$ $\therefore \lim_{x\to\infty} f(x) = e^0 = \boxed{1}$

$\boxed{66}$ $\lim_{x \to 0^+} (\sin x)(\ln x)$ $\quad 0 \cdot -\infty$

$$= \lim_{x \to 0^+} \frac{\ln x}{\csc x} \quad \frac{-\infty}{\infty}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}$$

$$= \lim_{x \to 0^+} \frac{1/x}{\frac{-1}{\sin x} \cdot \frac{1}{\tan x}}$$

$$= \lim_{x \to 0^+} \frac{-\sin x \tan x}{x} \quad \frac{0}{0}$$

Th.

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

$$= -\lim_{x \to 0^+} \frac{\sin x}{x} \cdot \lim_{x \to 0^+} \tan x$$

$$= -1 \cdot 0 = \boxed{0}$$

$\boxed{72}$ $\lim\limits_{x \to -\infty} \frac{2^x + 4^x}{5^x - 2^x}$ $\frac{0}{0}$ l'Hopital Rule

$$= \lim_{x \to -\infty} \frac{2^x \left[1 + \left(\frac{4}{2}\right)^x\right]}{2^x \left[\left[\frac{5}{2}\right]^x - 1\right]}$$

$$= \lim_{x \to -\infty} \frac{1 + \left(\frac{4}{2}\right)^x}{\left(\frac{5}{2}\right)^x - 1}$$

$$= \lim_{x \to -\infty} \frac{1 + 2^x}{\left(\frac{5}{2}\right)^x - 1} = \frac{1+0}{0-1} = \boxed{-1}$$

Another solution:-

$$\lim_{x \to -\infty} \frac{2^x + 4^x}{5^x - 2^x} = \lim_{x \to -\infty} \frac{2^x \ln 2 + 4^x \ln 4}{5^x \ln 5 - 2^x \ln 2}$$

$$= \lim_{x \to -\infty} \frac{\ln 2 + 2^x \ln 4}{\left(\frac{5}{2}\right)^x \ln 5 - \ln 2}$$

$$= \frac{\ln 2}{-\ln 2} = -1$$