Exercises 2.1 :

2.1.0 : True or False.

a. ① If x_n converges, then $\frac{x_n}{n}$ also converges ? True.

② If x_n converges to $a \in \mathbb{R}$ then $\frac{x_n}{n} \to 0$.

bdd يعني ① If x_n converges then there is $M>0$ s.t $|x_n| \leq M$

By Archimedian principle $K \in \mathbb{N}$ s.t $K > \frac{M}{\varepsilon} \to \frac{1}{K} < \frac{\varepsilon}{M} \to \frac{M}{K} < \varepsilon$

Then $n \geq K \to |x_n - n| \leq \frac{M}{n}$

$$\leq \frac{M}{K}$$

$$< \varepsilon \quad \blacksquare$$

نفس الإشي

② suppose that $x_n \to a$ as $n \to \infty$

since x_n converges then it is bdd (i.e, $\exists$ an $M>0$ s.t $|x_n| \leq M$, $\forall n \in \mathbb{N}$)

let $\varepsilon > 0$ be given we need to find $K \in \mathbb{N}$ s.t $n \geq K \to \left|\frac{x_n}{n} - 0\right| = \frac{|x_n|}{n} < \varepsilon$

use the Archimedian principle : $\exists K \in \mathbb{N}$ s.t $K > \frac{M}{\varepsilon}$: Then

$\to \varepsilon > \frac{M}{K}$

$n \geq K \to \left|\frac{x_n}{n} - 0\right| = \frac{|x_n|}{n}$

$\downarrow$

$\frac{1}{n} \leq \frac{1}{K}$

$$\leq \frac{|x_n|}{K}$$

$$\leq \frac{M}{K}$$

$$< \varepsilon$$

$\blacksquare$

2.1.0:

b. If x_n does not converge then $\frac{x_n}{n}$ doesn't converge ? False

$x_n = \sqrt{n}$ not converge But $\frac{x_n}{n} = \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$

c. If x_n converges and y_n is bounded then $x_n y_n$ Converges ? False

$x_n = 1$ converges and $y_n = (-1)^n$ is bounded But $x_n y_n = (-1)^n$ not converges.

d. If x_n converge to 0 and $y_n > 0$ for all $n \in \mathbb{N}$ then $x_n y_n$ converges ? False

$x_n = \frac{1}{n} \to 0$ and $y_n = n^2 > 0$ But $x_n y_n = n$ not converge.

2.1.1: prove that the following limits exist:

a. $2 - \frac{1}{n} \to 2$ as $n \to \infty$

let $\varepsilon > 0$ be given, we need to find $K \in \mathbb{N}$ s.t $n \geq K \to |\cancel{2} - \frac{1}{n} - \cancel{2}| < \varepsilon$

use Archimedian principle ($\exists n \in \mathbb{N}$ s.t $b < na$).

$\exists K \in \mathbb{N}$ s.t $K > \frac{1}{\varepsilon} \to \frac{1}{K} < \varepsilon$

$|\frac{1}{n}| < \varepsilon$

$\frac{1}{n} < \varepsilon$

$\boxed{n > \frac{1}{\varepsilon}}$

If $n \geq K$ then $|2 - \frac{1}{n} - 2| < \varepsilon$

$\frac{1}{n} \leq \frac{1}{K}$

$= \frac{1}{n}$

$\leq \frac{1}{K}$

$< \varepsilon$

2.1.1: b. $1+\frac{\pi}{\sqrt{n}} \to 1$ as $n \to \infty$

let $\varepsilon > 0$ be given, we need to find $K \in \mathbb{N}$ s.t $n \geq K \to \left|1+\frac{\pi}{\sqrt{n}}-1\right| < \varepsilon$

$\downarrow$

$\left|\frac{\pi}{\sqrt{n}}\right| < \varepsilon$

$\frac{\pi}{\sqrt{n}} < \varepsilon$

$\frac{\sqrt{n}}{\pi} > \frac{1}{\varepsilon}$

$\sqrt{n} > \frac{\pi}{\varepsilon}$

$n > \left(\frac{\pi}{\varepsilon}\right)^2$

use Archimedian principle : $\exists K \in \mathbb{N}$ s.t $K > \frac{\pi^2}{\varepsilon}$

$\sqrt{K} > \frac{\pi}{\varepsilon}$

$\frac{1}{\sqrt{K}} < \frac{\varepsilon}{\pi}$

If $n \geq K$ then $\left|1+\frac{\pi}{\sqrt{n}}-1\right| < \varepsilon$.

$\sqrt{n} \geq \sqrt{K}$

$\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{K}}$

$$= \left|\frac{\pi}{\sqrt{n}}\right| = \frac{\pi}{\sqrt{n}} \leq \frac{\pi}{\sqrt{K}} < \pi\left(\frac{\varepsilon}{\pi}\right) < \varepsilon \quad \blacksquare$$

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c. $3\left(1+\frac{1}{n}\right) \to 3$ as $n \to \infty$

let $\varepsilon > 0$ be given, we need to find $K \in \mathbb{N}$ s.t $n \geq K \to \left|3+\frac{3}{n}-3\right| < \varepsilon$

$\frac{3}{n} < \varepsilon$

$\frac{n}{3} > \frac{1}{\varepsilon}$

$\boxed{n > \frac{3}{\varepsilon}}$.

use Archimedian principle : $\exists K \in \mathbb{N}$ s.t $K > \frac{3}{\varepsilon}$

If $n \geq K$ then $\left|3+\frac{3}{n}-3\right| < \varepsilon$ $\qquad \frac{1}{K} < \frac{\varepsilon}{3}$

$\left(\frac{1}{n} \leq \frac{1}{K}\right) * 3$

$\frac{3}{n} \leq \frac{3}{K}$

$$= \frac{3}{n} \leq \frac{3}{K} < \cancel{3}\left(\frac{\varepsilon}{\cancel{3}}\right) < \varepsilon \quad \blacksquare$$

d. موجودة في المحاضرة قبل

2.1.2 : suppose that x_n is a sequence of real numbers that converges to 1 as $n \to \infty$.
prove that each of the following limits exists.

a. $1 + 2x_n \to 3$ as $n \to \infty$

let $\varepsilon > 0$ be given then we need find $K \in \mathbb{N}$ s.t $n \geq K \Rightarrow |1 + 2x_n - 3| < \varepsilon$

$|2x_n - 2| < \varepsilon$

$2|x_n - 2| < \varepsilon$

$|x_n - 1| < \frac{\varepsilon}{2}$.

Thus, $n \geq K \to |1 + 2x_n - 3|$

$= 2|x_n - 1|$

$< \cancel{2}\left(\frac{\varepsilon}{\cancel{2}}\right)$

$< \varepsilon$ $\boxed{}$

b. $\frac{(\pi x_n - 2)}{x_n} \to \pi - 2$ as $n \to \infty$

let $\varepsilon > 0$ be given then we need find $K \in \mathbb{N}$ s.t $n \geq K \Rightarrow \left|\frac{\pi x_n - 2}{x_n} - (\pi - 2)\right| < \varepsilon$

$\left|\frac{\pi x_n - 2}{x_n} - \frac{(\pi - 2)x_n}{x_n}\right| < \varepsilon$

$\left|\frac{\cancel{\pi x_n} - 2 - \cancel{\pi x_n} + 2x_n}{x_n}\right| < \varepsilon$

$\left|\frac{2x_n - 2}{x_n}\right| < \varepsilon$

$2\left|\frac{x_n - 1}{x_n}\right| < \varepsilon$

$\left|\frac{x_n - 1}{x_n}\right| < \frac{\varepsilon}{2}$

Thus $n \geq K \to \left|\frac{\pi x_n - 2}{x_n} - (\pi - 2)\right|$

$= 2\left|\frac{x_n - 1}{x_n}\right|$

$< \varepsilon$ $\boxed{}$

c. $\frac{(x_n^2 - e)}{x_n} \to 1-e$ as $n \to \infty$.

given $\varepsilon > 0$ there is an $K \in \mathbb{N}$ s.t $n \geq K$ implies $x_n > \frac{1}{2}$ and $|x_n - 1| < \frac{\varepsilon}{1+2e}$

Thus, $n \geq K$ and $\left| \frac{x_n^2 - e}{x_n} - (1-e) \right| = |x_n - 1| \left| 1 + \frac{e}{x_n} \right|$

$$\leq |x_n - 1| \left(1 + \frac{e}{|x_n|} \right)$$

$$< |x_n - 1| (1 + 2e)$$

$$< \frac{\varepsilon}{\cancel{1+2e}} \cancel{(1+2e)}$$

$$< \varepsilon$$

2.1.3 : Find two convergent subsequences that have different limits.

a. $3-(-1)^n$: if $n_K = 2K$, then $3-(-1)^{n_K} = 2$ converges to 2.

if $n_K = 2K+1$, then $3-(-1)^{n_K} = 4$ converges to 4.

b. $(-1)^{3n} + 2$: if $n_K = 2K$ then $(-1)^{3n_K} + 2 = (-1)^{6K} + 2 = 1 + 2 = 3$ converges to 3.

if $n_K = 2K+1$ then $(-1)^{3n_K} + 2 = (-1)^{6K+3} + 2 = -1 + 2 = 1$ converges to 1.

c. $\frac{(n - (-1)^n n - 1)}{n}$: if $n_K = 2K$, then $\left(\frac{n_K - (-1)^{n_K} n_K - 1}{n_K} \right) = \frac{-1}{2K}$ converges to 0.

if $n_K = 2K+1$, then $(\quad // \quad) = \frac{2n_K - 1}{n_K} = \frac{4K+1}{2K+1}$ converges to 2.

2.1.4: suppose that $x_n \in \mathbb{R}$

a. prove that $\{x_n\}$ is bounded iff there is a $C>0$ s.t $|x_n| \le C$ for all $n \in \mathbb{N}$.

supose x_n is bounded, By def, there are numbers M and m s.t $m \le x_n \le M$ $\forall n \in \mathbb{N}$

Set $C := \max\{1, |M|, |m|\}$, Then

$C>0$, $M \le C$, $m \ge -C$ Therefor $-C \le x_n \le C$

i.e $|x_n| < C$ $\forall n \in \mathbb{N}$.

Conversely: if $|x_n| < C$ $\forall n \in \mathbb{N}$ then x_n is bdd above by C and below by $-C$.

b. suppose that $\{x_n\}$ is bounded. prove that $\frac{x_n}{n^k} \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$

1.2.5: let C be a fixed, positive constant. If $\{b_n\}$ is a sequence of nonnegative numbers that converges to 0 and $\{x_n\}$ is a real sequence that satisfies $|x_n - a| \le C b_n$ for large n, prove that x_n converges to a.

let $\varepsilon > 0$ be given. since $b_n \to 0$ then $\exists K \in \mathbb{N}$ s.t $n \ge K \Rightarrow |b_n - 0| < \frac{\varepsilon}{C}$

$b_n < \frac{\varepsilon}{C}$

Hence, By hypothesis $n \ge K$ implise $|x_n - a| \le C b_n$

$$< C\left(\frac{\varepsilon}{C}\right)$$

$$< \varepsilon$$

Therefore, By def $x_n \to a$ as $n \to \infty$

1.2.6: let a be a fixed real number and define $x_n := a$ for $n \in \mathbb{N}$. prove that the constant sequence x_n converges.

If $x_n = a \ \forall n$, then $|x_n - a| = 0$ is less than any positive ε for all $n \in \mathbb{N}$.

Thus, By def $x_n \to a$ as $n \to \infty$

1.2.7: a. suppose that $\{x_n\}$ and $\{y_n\}$ converge to the same real number. prove that $x_n - y_n \to 0$ as $n \to \infty$.

let $\varepsilon > 0$ be given, since $x_n \to \beta$ and $y_n \to \beta$, By def $\exists K \in \mathbb{N}$ s.t

$n \ge K \Rightarrow |x_n - \beta| < \frac{\varepsilon}{2}$ and $|y_n - \beta| < \frac{\varepsilon}{2}$

By triangle inequality, $n \ge K$ implise

$$|(x_n - y_n) - 0| = |x_n - y_n| = |x_n - \beta + \beta - y_n|$$

$$\le |x_n - \beta| + |y_n - \beta|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

By def $x_n - y_n \to 0$ as $n \to \infty$

b. prove that the sequence $\{n\}$ does not converge.

By contradiction.

suppose $n \to \alpha$ as $n \to \infty$ for some $\alpha \in \mathbb{R}$.

given $\varepsilon = \frac{1}{2}$, $\exists K \in \mathbb{N}$ s.t $n \geq K \Rightarrow |n-\alpha| < \frac{1}{2}$

$$\to \quad 1 = |(n+1)-n| = |n+1-\alpha+\alpha-n|$$

?

$$\leq |n+1-\alpha| + |n-\alpha|$$

$$\leq |1+(n-\alpha)| + |n-\alpha|$$

$$< 1$$

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C. show that there exist unbounded sequences $x_n \neq y_n$ which satisfy the conclusion of part a.

let $x_n = n$ and $y_n = n + \frac{1}{n}$ Then $|x_n - y_n| = \frac{1}{n} \to 0$ as $n \to \infty$

But neither x_n nor y_n converges.

2.1.8: suppose that $\{x_n\}$ is a seq. in $\mathbb{R}$. prove that x_n converges to a iff every subsequence of x_n also converges to a.

By Thm

If $x_n \to a$ then $x_{nk} \to a$

Conversly, If $x_{nk} \to a$ for every subsequance then it converges for the subsequence x_n