Chapter 3. Second Order Linear Equations.

3.1 Homogeneous Equations with Constant coefficients.

3.3 Complex roots of the characterstic Equation

A second Order ODE has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \quad \cdots (1)$$

Eq. (1) is said to be Linear if f has the form: $f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)\,y \quad \cdots (2)$

i-e f is linear in y and y'

So, eq (1) can be written as:

$$y'' + p(t)\,y' + q(t)\,y = g(t) \quad \cdots (3)$$

If Eq (1) is not of the form of Eq. (3) then it is called Nonlinear.

Note: ① If $g(t) = 0$ in Eq.(3), then we call it Homogeneous Linear Equation.

② If $g(t) \neq 0$ in Eq.(3), then we call it Nonhomogeneous Linear Equation.

In sections 3.1, 3.3 and a part of 3.4, we will seek for the solution of the following 2nd order Linear Homogeneous equation with Constant coefficients

$$a y'' + b y' + c y = 0 \quad \ldots (4)$$

where a, b and c are constants.

To solve Eq.(4), we assume the solution as $y = e^{rt} \Rightarrow y' = r e^{rt}$ and $y'' = r^2 e^{rt}$

substitute in Eq.(4) we get:

$$ay'' + by' + cy = 0$$

$$\Leftrightarrow \quad ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$\Leftrightarrow \quad (ar^2 + br + c)\, e^{rt} = 0$$

$$\Leftrightarrow \quad ar^2 + br + c = 0$$ which is called

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the characteristic equation or auxiliary Eq.

Therefore, the roots of the Characteristic Eq. are

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So, we have 3 Cases to be Considered.

Case 1: r_1, r_2 are distinct real roots.

In this case: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

sec 3.4

Case 2: $r_1 = r_2 = r$ (repeated roots)

In this case: $y(t) = c_1 e^{rt} + c_2 t e^{rt}$

Case 3: r_1, r_2 are Conjugate Complex roots

$$r_1 = \alpha + \beta i \quad , \quad r_2 = \alpha - \beta i$$ sec. (3.3)

In this Case:

$$y(t) = c_1 e^{\alpha t} \cos\beta t + c_2 e^{\alpha t} \sin\beta t .$$

Example: Solve the following D.Es

① $y'' + 3y' + 2y = 0$

The characteristic Equation is:

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$\Rightarrow \quad r_1 = -1$ and $r_2 = -2$

$$\Rightarrow \quad y(t) = c_1 e^{-t} + c_2 e^{-2t}$$

② $y'' + 5y' + 6y = 0$

$$y(0) = 2 = y'(0) .$$

The characteristic equation is:

$$r^2 + 5r + 6 = 0$$

$$\Rightarrow \quad (r+2)(r+3) = 0$$

$$\Rightarrow \quad r_1 = -2 \quad , r_2 = -3$$

$$\Rightarrow \quad y(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

Now Using ICs:

$$y(0) = 2 \Rightarrow c_1 + c_2 = 2. \quad \cdots (*)$$

$$y'(t) = -2c_1 e^{-2t} + -3c_2 e^{-3t}$$

$$y'(0) = 2 \Rightarrow -2c_1 - 3c_2 = 2 \quad \cdots (**)$$

$$2\,Eq(*) + Eq(**) \Rightarrow -c_2 = 6 \Rightarrow \boxed{c_2 = -6}$$

Substitute in either (*) or (**), we get

$$-2c_1 - 3(-6) = 2 \Rightarrow -2c_1 = -16 \Rightarrow \boxed{c_1 = 8}$$

$$\Rightarrow \quad y(t) = 8e^{-2t} + -6e^{-3t}.$$

③ $y'' + 6y' + 9y = 0$

Charact. Eq. : $r^2 + 6r + 9 = 0$

$(r+3)(r+3) = 0$

$\Rightarrow \quad r_1 = r_2 = -3.$

$\Rightarrow \quad y(t) = c_1 e^{-3t} + c_2 t e^{-3t}.$

④ $y'' + y' + 9.25y = 0, \quad y(0) = 0, \quad y'(0) = 9.$

Charact. Eq. : $r^2 + r + 9.25 = 0$

$\Rightarrow \quad r_{1,2} = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(9.25)}}{2(1)}$

$\Rightarrow \quad r_{1,2} = \dfrac{-1 \pm \sqrt{-36}}{2} \Rightarrow r_{1,2} = -\frac{1}{2} \pm 3i$

$\Rightarrow \quad y(t) = c_1 e^{-\frac{1}{2}t} \cos 3t + c_2 e^{-\frac{1}{2}t} \sin 3t.$

Using I. Cs :

$y(0) = 0 \Rightarrow \boxed{0 = c_1}$

Since $c_1 = 0 \Rightarrow y(t) = c_2 e^{-\frac{1}{2}t} \sin 3t$.

$$y'(t) = 3c_2 e^{-\frac{1}{2}t} \cos 3t - \frac{1}{2} c_2 e^{-\frac{1}{2}t} \sin 3t.$$

$$\Rightarrow y'(0) = 9 \Rightarrow 9 = 3c_2 - 0 \Rightarrow \boxed{c_2 = 3}$$

$$\Rightarrow y(t) = 3e^{-\frac{1}{2}t} \sin 3t.$$

To discuss the Behavior of $y(t)$:

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} 3e^{-\frac{1}{2}t} \sin 3t = 0$$

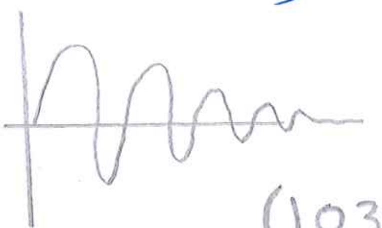
(Since $-1 \leq \sin 3t \leq 1$

$$\Rightarrow -3e^{-\frac{1}{2}t} \leq 3e^{-\frac{1}{2}t} \sin 3t \leq 3e^{-\frac{1}{2}t}$$

$$\underbrace{\lim_{t \to \infty} -3e^{-\frac{1}{2}t}}_{=0} \leq \lim_{t \to \infty} 3e^{-\frac{1}{2}t} \sin 3t \leq \underbrace{\lim_{t \to \infty} 3e^{-\frac{1}{2}t}}_{=0}$$

Using Sandwich thm, $\lim_{t \to \infty} 3e^{-\frac{1}{2}t} \sin 3t = 0$).

(Decay Oscillation).



(5) $16y'' - 8y' + 145y = 0$

$y(0) = 0$ and $y'(0) = 1$

The charact. Eq.

$$16r^2 - 8r + 145 = 0$$

$$\Rightarrow r = \frac{8 \pm \sqrt{64 - 4(16)(145)}}{2(16)}$$

$$= \frac{8 \pm \sqrt{64(1-145)}}{32} = \frac{8 \pm 8(12)i}{32}$$

$$r_{1,2} = \frac{1}{4} \pm 3i$$

$$\Rightarrow y(t) = c_1 e^{\frac{1}{4}t} \cos 3t + c_2 e^{\frac{1}{4}t} \sin 3t.$$

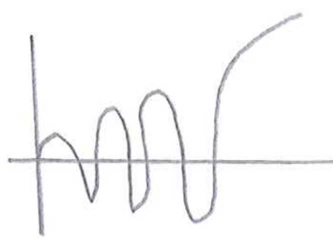
Using I.C: $y(0) = 0 \Rightarrow \boxed{0 = c_1}$

There fore: $y(t) = c_2 e^{\frac{1}{4}t} \sin 3t$

$$y'(t) = 3c_2 e^{\frac{1}{4}t} \cos 3t + \frac{1}{4} c_2 e^{\frac{1}{4}t} \sin 3t.$$

Using I.C: $y'(0) = 1 \Rightarrow 1 = 3c_2 \Rightarrow \boxed{c_2 = \frac{1}{3}}$

$$\Rightarrow y(t) = \frac{1}{3} e^{\frac{1}{4}t} \sin 3t.$$



Note that $\lim_{t \to \infty} y(t)$ is unbounded.

(Growing Oscillation).

Example: Let y be the solution of the IVP

$$y'' - y' - 2y = 0$$

$$y(0) = \alpha \quad \text{and} \quad y'(0) = 1$$

Find α for which $\lim_{t\to\infty} y(t) = 0$.

Sol: The charecteristic equation is

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2) = 0 \Rightarrow r_1 = 2, \; r_2 = -1$$

$$\Rightarrow \quad y(t) = c_1 e^{2t} + c_2 e^{-t}.$$

Using I.C: $y(0) = \alpha \Rightarrow \boxed{c_1 + c_2 = \alpha}$ ... ①

$$y'(t) = 2c_1 e^{2t} - c_2 e^{-t}$$

Using I.C: $y'(0) = 1 \Rightarrow \boxed{2c_1 - c_2 = 1}$ ... ②

$\therefore$ ① + ② $\Rightarrow 3c_1 = 1 + \alpha \Rightarrow c_1 = \frac{1+\alpha}{3}$.

① $\Rightarrow c_2 = \alpha - \frac{1+\alpha}{3} = \frac{3\alpha - 1 - \alpha}{3} = \frac{2\alpha - 1}{3}$

$$\therefore \; y(t) = \left(\frac{1+\alpha}{3}\right) e^{2t} + \left(\frac{2\alpha - 1}{3}\right) e^{-t}$$

Since $\lim_{t\to\infty} y(t) = 0$ and $e^{2t} \xrightarrow[t\to\infty]{} \infty$, then:

$$\frac{\alpha+1}{3} = 0 \Leftrightarrow \boxed{\alpha = -1}.$$

Home Work # 9 :

1) Let $y'' + y' - 2y = 0$, $y(0) = \beta$ and $y'(0) = 2$

Find the value(s) of β for which $\lim_{t\to\infty} y(t) = 0$

2) Consider $y'' + 2\alpha y' + y = 0$

Assume that the characterstic equation has Complex roots. Find the value of α for which

$\lim_{t\to\infty} y(t) = 0$

Euler's formula : $\boxed{e^{i\theta} = \cos\theta + i\sin\theta.}$ Proof Using Macclaurin series.

Example: Use Euler's formula to write the given expression in the form $a + bi$

① $e^{3i\pi} = e^{i(3\pi)} = \cos(3\pi) + i\sin(3\pi)$

$= -1 + i(0) = -1$

② $e^{2+\frac{\pi}{2}i} = e^2 \cdot e^{\frac{\pi}{2}i} = e^2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$= e^2(0 + i) = e^2 i.$

(3) $\pi^{-1+2i} = e^{(-1+2i)\ln\pi} = e^{-\ln\pi} \cdot e^{i(2\ln\pi)}$

$$= \frac{1}{\pi}\left(\cos(2\ln\pi) + i\sin(2\ln\pi)\right).$$

Example: Show that $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

Sol: $\dfrac{e^{ix} - e^{-ix}}{2i} = \dfrac{(\cos x + i\sin x) - (\cos(-x) + i\sin(-x))}{2i}$

even, odd

$$= \frac{\cancel{\cos x} + i\sin x - \cancel{\cos x} + i\sin x}{2i} = \sin x.$$

Note: In Case (3), Complex roots $r_{1,2} = \alpha + \beta i$

the solution is given by:

$$y(t) = c_1 e^{\alpha t}\cos t + c_2 e^{\alpha t}\sin t.$$

Since: $y_1(t) = e^{r_1 t} = e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$

$y_2(t) = e^{r_2 t} = e^{(\alpha - i\beta)t} = e^{\alpha t}(\cos(-\beta t) + i\sin(-\beta t))$

$\Rightarrow y(t) = c_1 y_1(t) + c_2 y_2(t) =$

$$= \underbrace{c_1 e^{\alpha t}(\cos\beta t}+ i\sin\beta t) + \underbrace{c_2 e^{\alpha t}(\cos\beta t} - i\sin\beta t)$$

$$= \underbrace{(c_1 + c_2)}_{D} e^{\alpha t}\cos\beta t + (\underbrace{c_1 i - c_2 i}_{C}) e^{\alpha t}\sin\beta t$$

$$= D e^{\alpha t}\cos\beta t + C e^{\alpha t}\sin\beta t.$$

(Q 20) page 144 , Find the solution of IVP

$$2y'' - 3y' + y = 0 \quad , y(0) = 2 \quad , y'(0) = \frac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

sol: The charact. Eq : $2r^2 - 3r + 1 = 0$

$$(2r - 1)(r - 1) = 0$$

Therefore $r_1 = \frac{1}{2}$, $r_2 = 1$

$\Rightarrow y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{t}$

Using I.C : $y(0) = 2 \Rightarrow 2 = c_1 + c_2$ ... (1)

$y'(t) = \frac{1}{2} c_1 e^{\frac{1}{2}t} + c_2 e^{t}$

Using I.C : $y'(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2} c_1 + c_2$ ... (2)

$\Rightarrow \boxed{c_1 = 3}$ and $\boxed{c_2 = -1}$

$\Rightarrow y(t) = 3e^{\frac{1}{2}t} - e^{t}.$

Maximum occurs at $y' = 0$ & $y'' < 0$

$y'(t) = \frac{3}{2} e^{\frac{1}{2}t} - e^{t} = 0 \Rightarrow \boxed{t = \ln \frac{9}{4}}$

Note that $y''(\ln \frac{9}{4}) < 0.$

$y(t) = 0 \iff 3e^{\frac{1}{2}t} - e^{t} = 0 \Rightarrow t = \ln 9$

(Q 34) page 166 (section 3.3)

Euler Equations: (Non Constant Coefficients).

An equation of the form :

$$A t^2 y'' + B t y' + C y = 0 \quad , \quad t > 0 \quad \cdots (*)$$

where $A, B, C \in \mathbb{R}$ is called Euler Equation.

To solve it Let $x = \ln t$ or $t = e^x$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot \left(\frac{1}{t}\right) \quad \cdots (1)$$

$$\frac{d^2y}{dt^2} = \frac{dy}{dx}\left(\frac{-1}{t^2}\right) + \frac{1}{t} \cdot \frac{d}{dt}\left(\frac{dy}{dx}\right)$$

$$= \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d^2y}{dx^2} \cdot \underbrace{\frac{dx}{dt}}_{= \frac{1}{t}} \quad \cdots (2)$$

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Now substitute (1) & (2) in Eq. (*) :

$$A t^2\left(\frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}\right) + B t\left(\frac{1}{t}\frac{dy}{dx}\right) + C y = 0$$

$$\Rightarrow \quad -A\frac{dy}{dx} + A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + C y = 0$$

$$\Rightarrow \quad A\frac{d^2y}{dx^2} + (B - A)\frac{dy}{dx} + C y = 0. \quad \cdots (**)$$

Notice that eq. (**) is a 2nd order Linear Homogeneous with Constant Coefficients

Therefore, the Characteristic equation is:

$$A r^2 + (B-A) r + C = 0$$

with $$r_{1,2} = \frac{-(B-A) \pm \sqrt{(B-A)^2 - 4AC}}{2A}$$

So:

1) If $r_1 \neq r_2 \in \mathbb{R}$, then:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \overset{y(t)}{=} c_1 e^{r_1 \ln t} + c_2 e^{r_2 \ln t}$$

$$y(t) = c_1 t^{r_1} + c_2 t^{r_2}.$$

2) If $r_1 = r_2 = r$, then

$$y(x) = c_1 e^{rx} + c_2 x e^{rx} = c_1 e^{r \ln t} + c_2 \ln t \, e^{r \ln t}$$

$$y(t) = c_1 t^r + c_2 (\ln t) t^r.$$

3) If $r_{1,2} = \alpha \pm \beta i$, then

$$y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

$$y(t) = c_1 t^{\alpha} \cos(\beta \ln t) + c_2 t^{\alpha} \sin(\beta \ln t).$$

(Q36) page 166. Solve the following D. E.

$$t^2 y'' + 4ty' + 2y = 0 \quad , \quad t > 0$$

Let $x = \ln t$, Using Eq. (**) with

$A = 1$, $B = 4$, $C = 2$, we have:

$$\frac{d^2y}{dx^2} + (4-1)\frac{dy}{dx} + 2y = 0$$

$$\Leftrightarrow \quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

Charact. Eq. : $r^2 + 3r + 2 = 0$

$$(r+1)(r+2) = 0$$

$\Rightarrow \quad r = -1$ and $r = -2$

$$\Rightarrow \quad y(x) = c_1 e^{-x} + c_2 e^{-2x}$$

$$\Rightarrow \quad y(t) = c_1 e^{-\ln t} + c_2 e^{-2\ln t}$$

$$= c_1 t^{-1} + c_2 t^{-2}$$

$$= \frac{c_1}{t} + \frac{c_2}{t^2} .$$

Example: Solve $4t^2y'' + 12ty' + 5y = 0, t>$

Sol: Let $x = \ln t$, then Using Eq. $(**)$:

with $A = 4$, $B = 12$, $C = 5$:

$$4\frac{d^2y}{dx} + (12-4)\frac{dy}{dx} + 5y = 0$$

$$\Rightarrow \quad 4\frac{d^2y}{dx} + 8\frac{dy}{dx} + 5y = 0$$

The charect. Eq. : $4r^2 + 8r + 5 = 0$

$$\Rightarrow \quad r_{1,2} = \frac{-8 \pm \sqrt{64 - 4(5)(4)}}{2(4)}$$

$$= \frac{-8 \pm 4i}{8} = -1 \pm \frac{1}{2}i$$

$$\Rightarrow \quad y(x) = c_1 e^{-x}\cos\left(\tfrac{1}{2}x\right) + c_2 e^{-x}\sin\left(\tfrac{1}{2}x\right)$$

$$= c_1 e^{-\ln t}\cos\left(\tfrac{1}{2}\ln t\right) + c_2 e^{-\ln t}\sin\left(\tfrac{1}{2}\ln t\right)$$

$$= \frac{c_1}{t}\cos\left(\tfrac{1}{2}\ln t\right) + \frac{c_2}{t}\sin\left(\tfrac{1}{2}\ln t\right)$$

3.2 Solutions of Linear Homogeneous Equations ; The Wronskian.

Theorem (3.2.1): (Existence and Uniqueness Thm):

Consider the IVP : $\begin{cases} y'' + p(t)\,y' + q(t)\,y = g(t) \\ y(t_0) = y_0 \ , \ y'(t_0) = y'_0 \end{cases}$

where p, q and g are continuous on an open interval $I = (\alpha, \beta)$, that contains t_0, then there is exactly one solution for the IVP.

Example: Find the largest interval in which the solution of the IVP is certain to exist.

$$(t^2 - 3t)\,y'' + t\,y' - (t+3)\,y = 0$$

$$y(1) = 2 \ , \ y'(1) = 1$$

Sol:
$$y'' + \frac{t}{t(t-3)}\,y' - \frac{(t+3)}{t(t-3)}\,y = 0$$

$$p(t) = \frac{t}{t(t-3)} \ , \ q(t) = \frac{-(t+3)}{t(t-3)} \ , \ g(t) = 0$$

p, q and g are continuous on $(-\infty,0)\cup(0,3)\cup(3,\infty)$

So, the largest interval that contains $t_0 = 1$ is

$(0,3)$. in which the solution is certain to exist.

Linear operator:

Define $L[y] = y'' + p(t)\, y' + q(t)\, y = 0$ ... (1)

If y_1 is a solution of the D.E, then $L[y_1] = 0$.

(i.e) $L[y_1] = y_1'' + p(t)\, y_1' + q(t)\, y_1 = 0$.

Theorem (3.2.2): (principle of Superposition).

If y_1 and y_2 are two solutions of Eq (1),

$$L[y] = y'' + p(t)\, y' + q(t)\, y = 0.$$

then the linear combination $c_1 y_1 + c_2 y_2$ is

also a solution for any values of c_1 and c_2.

Proof: Suppose $L[y_1]=0$ and $L[y_2]=0$

need to show $L[c_1y_1+c_2y_2]=0$?

$$L[c_1y_1+c_2y_2] = (c_1y_1+c_2y_2)'' + p(t)(c_1y_1+c_2y_2)' + q(t)(c_1y_1+c_2y_2)$$

$$= c_1y_1'' + c_2y_2'' + p(t)(c_1y_1') + p(t)(c_2y_2') + q(t)c_1y_1 + q(t)c_2y_2$$

$$= c_1\left(y_1'' + p(t)y_1' + q(t)y_1\right) + c_2\left(y_2'' + p(t)y_2' + q(t)y_2\right)$$

$$= c_1L[y_1] + c_2L[y_2] = c_1\cdot 0 + c_2\cdot 0 = 0$$

$\Rightarrow L[c_1y_1+c_2y_2]=0$ (i.e) $c_1y_1+c_2y_2$ is a solution

Note: We call $c_1y_1+c_2y_2$ a General Solution.

Def: The Wronskian of the solutions y_1 and y_2

is given by: $W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$

Example: Find $W(t, t\ln t)$, $t>0$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t\ln t \\ 1 & 1+\ln t \end{vmatrix}$$

$$= t(1+\ln t) - t\ln t = t$$

Theorem (3.2.3) Suppose that y_1 and y_2 are two solutions of the IVP:

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$$L[y] = y'' + p(t)y' + q(t)y = 0 \quad , \quad y(t_0) = y_0 \, , \, y'(t_0) = y_0'$$

Then by thm (3.2.2), it is always possiple to choose the constants c_1 and c_2 so that $c_1 y_1 + c_2 y_2$ satisfies the D.E iff $W(y_1, y_2) = y_1 y_2' - y_1' y_2 \neq 0$ at $t_0 \in I$.

Proof: By thm (3.2.2), we can find c_1 and c_2 such that $y(t) = c_1 y_1 + c_2 y_2$ is a solution.

Using I.C : $y(t_0) = y_0 \Rightarrow$ and : $y'(t_0) = y_0' \Rightarrow$

$$\begin{cases} y_0 = c_1 y_1(t_0) + c_2 y_2(t_0) \\ y_0' = c_1 y_1'(t_0) + c_2 y_2'(t_0) \end{cases}$$

$\downarrow$

Solve the System

Using Cramer's Method, we can find c_1 and c_2

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \quad \text{and} \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

So, for c_1 and c_2 to make sence, we should have

$$W(y_1, y_2)(t_0) \neq 0.$$

Theorem (3.2.4): Suppose that y_1 and y_2 are two solutions of the D.Es.

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0. \quad \cdots (3)$$

Then the family of solutions

$$y = c_1 y_1 + c_2 y_2, \quad c_1, c_2 \text{ arbitrary.}$$

includes all solutions of Eq (3) if and only if there is a point t_0 when $W(y_1, y_2)(t_0) \neq 0$.

Def: We say that y_1 and y_2 are Linearly independent on I, if $c_1 y_1 + c_2 y_2 = 0$ Implies $c_1 = c_2 = 0$

Example: $f(x) = \sin 2x$, $g(x) = \sin x \cos x$

If $c_1 \sin 2x + c_2 \sin x \cos x = 0$, then We can find $c_1 = 1$ and $c_2 = -2$ such that

$$1(\sin 2x) - 2 \sin x \cos x = 0$$

then $f(x)$ and $g(x)$ are Linearly dependent.

Def: We say that y_1 and y_2 are Linearly Independent on I iff $W(y_1, y_2)(t) \neq 0$, for at least one $t \in I$.

Example: $y_1 = e^{2t}$ and $y_2 = e^{3t}$ are L.I on $(-\infty, \infty)$ since:

Linearly Independent

$$W(y_1, y_2) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = 3e^{5t} - 2e^{5t} = e^{5t} \neq 0, \forall t \in \mathbb{R}$$

Example: Are $\{1, x, x^2\}$ Linearly Independent?

$$W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

$\therefore$ $\{1, x, x^2\}$ are Linearly independent on $\mathbb{R}$.

Remark: If y_1 and y_2 are Linearly dependent, then $W(y_1, y_2) = 0$.

But the Converse is Not true. (i.e) if $W(y_1, y_2) = 0$, we don't know if y_1, y_2 are Linearly Independent or not, we need to check!

Def: (Fundemental set of Solutions)

The Solutions y_1 and y_2 are said to form a fundemental set of solutions, (FSS), of the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

iff $W(y_1, y_2) \neq 0$.

Example: Show that $y_1 = \sqrt{t}$ and $y_2 = \frac{1}{t}$ form a fundamental set of solution for

$$2t^2 y'' + 3ty' - y = 0 \quad , t > 0$$

Sol: (1) $y_1 = \sqrt{t}$ is a solution of the D.E, since

$$2t^2\left(\frac{-1}{4t^{3/2}}\right) + 3t\left(\frac{1}{2\sqrt{t}}\right) - \sqrt{t}$$

$$= -\frac{1}{2} t^{2-3/2} + \frac{3}{2} t^{1-\frac{1}{2}} - \sqrt{t}$$

$$= -\frac{1}{2} t^{\frac{1}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0$$

Similarly, $y_2 = \frac{1}{t}$ is also a solution, since:

$$2t^2\left(\frac{2}{t^3}\right) + 3t\left(\frac{-1}{t^2}\right) - \frac{1}{t} = 0 .$$

(2) $$W\left(\sqrt{t}, \frac{1}{t}\right) = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2} t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -\frac{3}{2} \cdot \frac{1}{\sqrt{t^3}} \neq 0$$

$\Rightarrow \left\{\sqrt{t}, \frac{1}{t}\right\}$ are Linearly Independent.

Therefore $\left\{\sqrt{t}, \frac{1}{t}\right\}$ Form a FSS.

Example: Find FSS of $3y'' + y' - 2y = 0$.

Sol: characteristic equ. $3r^2 + r - 2 = 0$

$\Rightarrow (3r-2)(r+1) = 0 \Rightarrow r_1 = \frac{2}{3}$ & $r_2 = -1$

$\therefore y_1 = e^{\frac{2}{3}t}$, $y_2 = e^{-t}$.

$$W(y_1, y_2) = \begin{vmatrix} e^{\frac{2}{3}t} & e^{-t} \\ \frac{2}{3}e^{\frac{2}{3}t} & -e^{-t} \end{vmatrix} = -\frac{5}{3}e^{-\frac{1}{3}t} \neq 0 \ , \ \forall t \in \mathbb{R}$$

$\Rightarrow \{e^{\frac{2}{3}t}, e^{-t}\}$ Form a FSS.

Home work # 10

(1) Check if $y_1 = x$ and $y_2 = xe^{x}$ form a FSS' for the D.E

$$x^2 y'' - x(x+2)y' + (x+2)y = 0 \ , \ x > 0$$

(2) Check if $y_1 = x$, $y_2 = \sin x$ form a FSS for the D.E

$$(1 - x\cot x)y'' - xy' + y = 0 \ , \ 0 < x < \pi.$$

Theorem (3.2.5): Consider the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on some open interval I. Choose a point $t_0 \in I$.

Let y_1 be the solution of $L[y]=0$ and satisfies

$$y_1(t_0) = 1 \quad , \quad y_1'(t_0) = 0 \quad , \text{ and let}$$

y_2 be another solution of $L[y]=0$ and satisfies

$$y_2(t_0) = 0 \quad , \quad y_2'(t_0) = 1 \quad , \text{ then}$$

y_1 and y_2 form a fundemental set of solutions of $L[y]=0$.

Remark: $W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

Hence by this theorem & thm (3.2.1) we observe that the existence of the functions y_1 and y_2 is ensured and that they form a fundemental set of solutions.

Example: Find the FSS specified by thm 3.2.5

for $y'' - y = 0$ Using $t_0 = 0$.

Sol: Charac. equation : $r^2 - 1 = 0 \Longleftrightarrow r_1 = 1$ & $r_2 = -1$

$\therefore\ y_1 = e^t$ and $y_2 = e^{-t}$

$\therefore\ y(t) = c_1 e^t + c_2 e^{-t}$

Using $\left.\begin{array}{l} y(0) = 1 \Rightarrow c_1 + c_2 = 1 \\ y'(0) = 0 \Rightarrow c_1 - c_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{2} \end{array}$

$\therefore\ y_3 = \frac{1}{2} e^t + \frac{1}{2} e^{-t} = \cosh t.$

Now: $\left.\begin{array}{l} y(0) = 0 \Rightarrow c_1 + c_2 = 0 \\ y'(0) = 1 \Rightarrow c_1 - c_2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = \frac{1}{2} \\ c_2 = -\frac{1}{2} \end{array}$

$\therefore\ y_4 = \frac{1}{2} e^t - \frac{1}{2} e^{-t} = \frac{e^t - e^{-t}}{2} = \sinh t$

$\therefore\ \{y_3, y_4\} = \{\cosh t, \sinh t\}$ form a Fundamental set of solutions.

Theorem (3.2.6) (Abel's theorem):

If y_1 and y_2 are two solutions of the d.e.

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0 ,$$

where p and q are contineons on an open Interval I

then $W(y_1, y_2) = C\, e^{-\int p(t)\,dt}$, where C is

constant that depends on y_1 and y_2 , but

not on t.

Moreover, If $c = 0$, then $W(y_1, y_2) = 0$, $\forall t \in I$.

If $c \neq 0$, $W(y_1, y_2) \neq 0$, $\forall t \in I$.

Proof: Since y_1 and y_2 are solutions of $L[y] = 0$

then

$$y_1'' + p(t)\,y_1' + q(t)\,y_1 = 0 \qquad \cdots (1)$$

and

$$y_2'' + p(t)\,y_2' + q(t)\,y_2 = 0 \qquad \cdots (2)$$

Multiply Eq.(1) by $-y_2$, and Eq.(2) by $+y_1$

then add the resulting equations, we obtain:

$$\oplus \begin{cases} -y_2\, y_1'' - p(t)\, y_2\, y_1' - q(t)\, y_1 y_2 = 0 \\ y_1\, y_2'' + p(t)\, y_1\, y_2' + q(t)\, y_1 y_2 = 0 \end{cases}$$

$$\underbrace{y_1 y_2'' - y_2 y_1''}_{W'} + p(t)\underbrace{(y_1 y_2' - y_2 y_1')} = 0 \quad \cdots (3)$$

Notice that $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (y_1 y_2' - y_2 y_1')$

$$W'(y_1, y_2) = y_1 y_2'' + \cancel{y_1' y_2'} - y_2 y_1'' - \cancel{y_2' y_1'}$$

$$W' = y_1 y_2'' - y_2 y_1''$$

$\therefore$ Eq. (3) becomes : $W' + p(t)\, W = 0$, (seperable).

$\Rightarrow \dfrac{dW}{W} = -p(t)\, dt \Rightarrow \int \dfrac{dW}{W} = -\int p(t)\, dt$

$\Rightarrow \ln|W| = -\int p(t)\, dt + C_1$

$\Rightarrow W = \pm e^{C_1} e^{-\int p(t)\, dt}$

$\Rightarrow W = C\, e^{-\int p(t)\, dt}$, where C is Constant

Since $e^{-\int p(t) dt} \neq 0$, then $W(y_1, y_2) \neq 0$ unless $C = 0$.

Example: Find the Wronskian of two solutions

of $t^2 y'' - t(t+2)y' + (t+2)y = 0 \ , \ t > 0$

Sol: $y'' - \left(\frac{t+2}{t}\right) y' + \left(\frac{t+2}{t^2}\right) y = 0$.

$p(t) = -\left(\frac{t+2}{t}\right) = -1 - \frac{2}{t}$

$\therefore W(t) = C e^{-\int p(t)\,dt} = C e^{\int 1 + \frac{2}{t}\,dt} = C e^{t + 2\ln t}$

$= C t^2 e^t . \quad t > 0 .$

Example: Suppose that $y_1 = \sqrt{t}$ is a solution of, $t>0$

$2t^2 y'' + 3t y' - y = 0$. Find 2nd Independent Solution.

Sol: $W(y_1, y_2) = \begin{vmatrix} \sqrt{t} & y_2 \\ \frac{1}{2\sqrt{t}} & y_2' \end{vmatrix} = \sqrt{t}\, y_2' - \frac{y_2}{2\sqrt{t}}$.

And $W(y_1, y_2) = C e^{-\int p(t)\,dt} = C e^{-\int \frac{3}{2} \cdot \frac{1}{t}\,dt}$

$\therefore W(y_1, y_2) = C e^{-\frac{3}{2}\ln t} = C t^{-\frac{3}{2}} = \frac{C}{t^{3/2}}$.

$$\therefore \quad \sqrt{t}\, y_2' - \frac{1}{2\sqrt{t}}\, y_2 = \frac{c}{t^{3/2}}$$

$$\Rightarrow \quad y_2' - \frac{1}{2t}\, y_2 = \frac{c}{t^2}, \quad \text{(1st Order Linear)}$$

Using Integrating factor :

$$\mu(t) = e^{\int \frac{-1}{2t} dt} = e^{-\frac{1}{2}\ln t} = t^{-\frac{1}{2}} = \frac{1}{\sqrt{t}}$$

$$\therefore \quad y_2(t) = \frac{1}{\mu(t)} \left[\int \mu(t)\, g(t)\, dt + c' \right]$$

$$= \sqrt{t} \left[\int \frac{1}{\sqrt{t}} \cdot \frac{c}{t^2}\, dt + c' \right]$$

$$= \sqrt{t} \left[\underbrace{c\left(-\frac{2}{3}\right)}_{A} t^{-\frac{3}{2}} + c' \right]$$

$$= \sqrt{t} \left[A\, t^{-\frac{3}{2}} + c' \right]$$

$$y_2(t) = A\, t^{-1} + c' \sqrt{t}.$$

Remark : Using Abel's theorem, we can find the Wronskian of any two solutions of the differential equation, without knowing the two solutions.

Thm: Consider the equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0, \quad \text{where}$$

p and q are continuous functions.

If $y = u(t) + i\,v(t)$ is a complex solution of the D.E, then $u(t)$ and $v(t)$ are two solutions of the D.E.

Proof: $L[y] = L[u(t) + i\,v(t)] = 0$ (complex solution)

$$= \left[u''(t) + i\,v''(t)\right] + p(t)\left[u'(t) + i\,v'(t)\right] + q(t)\left[u(t) + i\,v(t)\right]$$

$$= \left[u''(t) + p(t)u'(t) + q(t)u(t)\right] + i\left[v''(t) + p(t)v'(t) + q(t)v(t)\right]$$

$$= L[u(t)] + i\,L[v(t)] = 0$$

$$\Longleftrightarrow \quad L[u(t)] = L[v(t)] = 0.$$

Q34) (page 157). If $ty'' + 2y' + te^{t}y = 0$

has y_1 and y_2 as a fundemental set of solutions

and if $W(y_1, y_2)(1) = 2$. Find $W(y_1, y_2)(5)$.

sol: $y'' + \frac{2}{t}y' + e^{t}y = 0$

$$W(y_1, y_2) = c\, e^{-\int \frac{2}{t}dt} = c\, e^{-2\ln|t|} = c(t)^{-2}.$$

Now, $W(y_1, y_2)(1) = 2 \Rightarrow 2 = c(1)^{-2}$

$$\Rightarrow \boxed{c = 2}$$

$\therefore W(y_1, y_2)(t) = 2(t^{-2})$

$\therefore W(y_1, y_2)(5) = 2(5)^{-2} = \frac{2}{25}$.

3.4 Repeated Roots, Reduction of Order.

Reduction of Order Method.

Consider the differential equation

$$y'' + p(t)\,y' + q(t)\,y = 0 \qquad \ldots (*)$$

Suppose we know one Solution $y_1(t)$ of Eq. $(*)$.

To find a second L.I Solution for Eq. $(*)$, we let

$$\boxed{y_2(t) = v(t)\,y_1(t)} \qquad \ldots (\mathrm{I})$$

, then

$$y_2'(t) = v(t)\,y_1'(t) + v'(t)\,y_1(t)$$

$$y_2''(t) = v(t)\,y_1''(t) + v'(t)\,y_1'(t) + v'(t)\,y_1'(t) + v''(t)\,y_1(t)$$

$$y_2''(t) = v(t)\,y_1''(t) + 2v'(t)\,y_1'(t) + v''(t)\,y_1(t)$$

substitute y_2, y_2' & y_2'' in Eq. $(*)$ we get:

$$\left[v\,y_1'' + 2v'y_1' + v''y_1\right] + p(t)\left[v\,y_1' + v'y_1\right] + q(t)\left[v\,y_1\right] = 0$$

$$\Rightarrow \quad v\underbrace{\left[y_1'' + p(t)\,y_1' + y_1\right]}_{\substack{\parallel \\ 0}} + v'\left[2y_1' + p(t)y_1\right] + v''y_1 = 0$$

0, since y_1 is a Solution of Eq. $(*)$

$$\Rightarrow \quad y_1 v'' + [2y_1' + p(t)y_1]v' = 0 \quad \dots (**)$$

Let $\omega = v'$, then $\omega' = v''$, then $(**)$ becomes

$$y_1 \omega' + (2y_1' + p(t)y_1)\omega = 0 \quad \dots (***)$$

Eq. $(***)$ is 1st order linear ODE.

$$(***) \Rightarrow \omega' + \left(\frac{2y_1'}{y_1} + p(t)\right)\omega = 0 \quad , y_1 \neq 0.$$

$$\mu(t) = e^{\int\left(\frac{2y_1'}{y_1} + p(t)\right)dt} = e^{2\ln|y_1| + \int p(t)\,dt}$$

$$= y_1^2 \, e^{\int p(t)\,dt}$$

$$\therefore \omega(t) = \frac{1}{y_1^2} e^{-\int p(t)dt}\left[\int 0 \cdot \left(y_1^2 e^{\int p(t)dt}\right)dt + C\right]$$

$$\therefore \omega(t) = \frac{C\, e^{-\int p(t)\,dt}}{y_1^2} = \frac{W(y_1, y_2)}{y_1^2}$$

$$\therefore v'(t) = \frac{W(y_1, y_2)}{y_1^2}$$

From Eq. (I):

$$\therefore v(t) = \frac{y_2}{y_1} = \int \frac{W(y_1, y_2)}{y_1^2}\,dt.$$

$$\Rightarrow \boxed{y_2 = y_1 \int \frac{W(y_1, y_2)}{y_1^2}\, dt.} \quad \text{(Reduction Formula)}$$

Example: Given that $y_1 = t^2$ is a solution of

$$t y'' - 6y' + \frac{10}{t} y = 0 \quad , t > 0 .$$

Use the method of Reduction of order to find a second Linearly Independent solution of the given D.E.

sol: Let $y_2 = v\, y_1 = t^2 v(t)$

$$y_2' = t^2 v'(t) + 2t\, v(t)$$

$$y_2'' = t^2 v''(t) + 2t v'(t) + 2t\, v'(t) + 2\, v(t)$$

$$y_2'' = t^2 v''(t) + 4t v'(t) + 2v(t)$$

substitute y_2, y_2' and y_2'' in the D.E, we get:

$$t\left(t^2 v''(t) + 4t\, v'(t) + 2v(t)\right) - 6\left(t^2 v'(t) + 2t v(t)\right) + \frac{10}{t}\left(t^2 v(t)\right)$$

$$\Rightarrow t^3 v''(t) + \underbrace{4t^2}\, v'(t) + 2t\, v(t) \underbrace{- 6t^2} v'(t) - 12t\, v(t) + 10\, t\, v(t)$$

$$\Rightarrow t^3 v''(t) - 2t^2 v'(t) = 0$$

$$\Rightarrow v''(t) - \frac{2}{t} v'(t) = 0 \quad , t > 0.$$

Let $\omega = v'$, then $\omega' = v''$,

$\Rightarrow \quad \omega' - \frac{2}{t}\omega = 0 \quad , t>0$

$\mu(t) = e^{\int -\frac{2}{t} dt} = e^{-2\ln t} = t^{-2} \quad , t>0$

$\Rightarrow \omega(t) = \frac{1}{t^{-2}}\left[\int t^{-2}(0)\,dt + C\right]$

$\omega(t) = C t^2$

$\Rightarrow \quad v'(t) = C t^2 \Rightarrow v(t) = \frac{C t^3}{3} + D$

$\therefore \quad v(t) = A t^3 + D \quad$, where $A = \frac{C}{3}$.

$\therefore \quad y_2(t) = (At^3 + D) y_1 = (At^3 + D) t^2$

$\therefore \quad y_2(t) = A t^5 + D t^2$

$\therefore \quad y_2(t) = A t^5$

Example: Use the Reduction Formula to find y_2 in the previous example.

sol: $y'' - \frac{6}{t} y' + \frac{10}{t^2} y = 0 \quad , t > 0.$

$$W(y_1, y_2) = c\, e^{-\int -\frac{6}{t} dt} = c\, e^{6 \ln|t|} = c\, t^6 , \; t>0$$

$$\therefore \; y_2 = y_1 \int \frac{W(y_1, y_2)}{y_1^2} dt = t^2 \int \frac{c\, t^6}{t^4} dt$$

$$y_2 = t^2 \left(c\, \frac{t^3}{3} \right) = \frac{c}{3} t^5 = A t^5 .$$

Home Work #11.

(1) Given that $y_1 = \frac{1}{t}$ is a Solution of

$$2t^2 y'' + 3t y' - y = 0 \quad , \; t > 0.$$

Use the Method of Reduction of order to find y_2?

(2) Given that $y_1 = \frac{\sin x}{\sqrt{x}}$ is one Solution of

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4} \right) y = 0 \quad , \; x > 0 .$$

Find y_2 Using the Reduction Formula.

3.5 Nonhomogeneous Equations, Method of undetermined Coefficients.

Consider the nonhomogeneous differential equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = g(t) \quad \ldots (1)$$

where p, q and g are Continuous functions on an open Interval I. The Corresponding homogeneous differential equation of Eq. (1) is

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0 \,. \quad \ldots (2)$$

Theorem 3.5.1: ① If Y_1 and Y_2 are two Solutions of Eq. (1), then $Y_1 - Y_2$ is a Solution of Eq. (2)

② If y_1, y_2 are fundemental set of Solutions of Eq. (2), then $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$.

proof: ① Since Y_1 and Y_2 are two Solutions of Eq. (1), then: $L[Y_1] = g(t)$ and $L[Y_2] = g(t)$

$$\Rightarrow \; L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0.$$

Therefore $Y_1 - Y_2$ is a Solution of Eq. (2).

② Since $Y_1 - Y_2$ is a solution of Eq. (2), and

y_1 and y_2 are fundemental set of solutions of Eq. (2),

then $Y_1 - Y_2$ can be written as a linear combination

of y_1 and y_2, (i-e) $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$

Example: prove that if Y_1, Y_2 are solutions of

$L[y] = g(t)$, then $\frac{1}{4} Y_1 + \frac{3}{4} Y_2$ is also a solution

of $L[y] = g(t)$.

Proof: $L\left[\frac{1}{4} Y_1 + \frac{3}{4} Y_2\right] = \frac{1}{4} \underbrace{L[Y_1]} + \frac{3}{4} \underbrace{L[Y_2]}$

$= \frac{1}{4} g(t) + \frac{3}{4} g(t)$

$= g(t)$

Y_1 & Y_2 are two solutions of $L[y] = g(t)$.

Method of Undetermined Coefficients:

Consider the nonhomogeneous 2nd order Linear D.E

$$a y'' + b y' + c y = g(t) \quad \cdots (3).$$

where a, b, c are constants, and $g(t)$ is one of the following:

1) $g(t)$ is a polynomial

2) $g(t)$ is an exponential function (ex. e^{rt}).

3) $g(t)$ is sin or Cosine functions, ($\sin\beta t$ or $\cos\beta t$)

4) $g(t)$ is a finite sums and products of (1), (2) and (3).

Remark : This method is limited to Linear D.E (3) where the Conditions on a, b, c and $g(t)$ as above.

Now, to solve Eq. (3) by this method, we must do the following:

1. Find y_h = the Solution of the corresponding homogeneous equation : $a y'' + b y' + c y = 0$

2. Find a particular Solution y_p of the nonhomogeneous Eq. (3). Note that y_p depends totally on the form of $g(t)$ as follows:

i) If $g(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$, poly

then we let $y_p = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$.

We substitute y_p in Eq.(3) to find $A_0, A_1, \ldots, A_n$.

Then check that y_p is Linearly Independent to y_h.

If Not, we multiply by t^s with $s = 0, 1$ or 2.

Example: Find y_p for $y'' - y' - 6y = 1 - 3x^2$.

y_h : $r^2 - r - 6 = 0 \iff (r+2)(r-3) = 0 \iff r = -2, 3$

$\therefore \; y_h(t) = c_1 e^{3x} + c_2 e^{-2x}$

Now: Let $y_p = A_0 + A_1 x + A_2 x^2$

$$y_p' = A_1 + 2A_2 x$$

$$y_p'' = 2A_2$$

$\Rightarrow \; 2A_2 - (A_1 + 2A_2 x) - 6(A_0 + A_1 x + A_2 x^2) = 1 - 3x^2$

$\Rightarrow \; (2A_2 - A_1 - 6A_0) + (-2A_2 - 6A_1)x - 6A_2 x^2 = 1 - 3x^2$

$$\Rightarrow \left.\begin{aligned} 2A_2 - A_1 - 6A_0 &= 1 \\ -2A_2 - 6A_1 &= 0 \\ -6A_2 &= -3 \end{aligned}\right\} \Rightarrow \begin{aligned} A_2 &= \frac{-3}{-6} = \frac{1}{2} \\ A_1 &= -\frac{1}{6} \\ A_0 &= \frac{1}{36} \end{aligned}$$

$\therefore \; y_p = \frac{1}{2}x^2 - \frac{1}{6}x + \frac{1}{36}.$

Note that y_h and y_p are Linearly Independent, therefore, the General Solution is given by:

$$y(x) = y_h + y_p = C_1 e^{3x} + C_2 e^{-2x} + \frac{1}{2}x^2 - \frac{1}{6}x + \frac{1}{36}.$$

Example: Solve $y'' = 3t^2$.

For y_h: $r^2 = 0 \Rightarrow r_{1,2} = 0$

$\therefore \; y_h(t) = C_1 + C_2 t.$

Now, Let $y_p(t) = (At^2 + Bt + C)\,t^2 = At^4 + Bt^3 + Ct^2.$

$y_p' = 4At^3 + 3Bt^2 + 2Ct \quad , \; y_p'' = 12At^2 + 6Bt + 2C$

$\Rightarrow \; 12At^2 + 6Bt + 2C = 3t^2 \Rightarrow \begin{cases} C = 0 \; , \; B = 0 \\ A = \frac{1}{4} \end{cases}$

$\therefore \; y_p(t) = \frac{1}{4}t^4.$

$\therefore$ General Solution is $y(t) = C_1 + C_2 t + \frac{1}{4}t^4.$

ii) If $g(t) = C_1 e^{at}$, then we let $y_p = A e^{at}$

We substitute y_p in the D.E, then find A

such that y_p is L.I with y_h.

Otherwise, we multiply by t^s, with $s = 0, 1, 2$.

Example: Solve $y'' - 3y' - 4y = 3e^{2t}$

sol: y_h : $r^2 - 3r - 4 = 0 \iff (r+1)(r-4) = 0$

$\therefore r = -1$ & $r = 4 \Rightarrow y_h(t) = c_1 e^{-t} + c_2 e^{4t}$

Now, Let $\underbrace{y_p = A e^{2t}}_{y_p \text{ \& } y_h \text{ are L.I}} \Rightarrow y_p' = 2Ae^{2t} \Rightarrow y_p'' = 4Ae^{2t}$

$\Rightarrow 4Ae^{2t} - 3(2Ae^{2t}) - 4(Ae^{2t}) = 3e^{2t}$

$\Rightarrow 4A - 6A - 4A = 3 \Rightarrow -6A = 3 \Rightarrow \boxed{A = \frac{-1}{2}}$

$\therefore y_p(t) = -\frac{1}{2} e^{2t}$

$\therefore$ General Solution is

$y(t) = y_h + y_p = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{2} e^{2t}$.

Example : Solve $y'' - 3y' - 4y = 3e^{2t}$

with $y(0) = 0$ & $y'(0) = 2$

Sol : $y(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{2} e^{2t}$ (previous Example)

$\Rightarrow y(0) = 0 \Rightarrow 0 = c_1 + c_2 - \frac{1}{2}$

$y'(0) = 2$: $y'(t) = -c_1 e^{-t} + 4c_2 e^{4t} - e^{2t}$

$\therefore y'(0) = 2 \Longrightarrow 2 = -c_1 + 4c_2 - 1$

solving the two equations : $c_1 = -\frac{1}{5}$, $c_2 = \frac{7}{10}$

$\therefore y(t) = -\frac{1}{5} e^{-t} + \frac{7}{10} e^{4t} - \frac{1}{2} e^{2t}$

Example! Solve $y'' - y' - 6y = 7e^{3x}$

Sol: $r^2 - r - 6 = 0 \Leftrightarrow (r+2)(r-3) = 0 \Leftrightarrow r_1 = 3, r_2 = -2$

$\therefore y_h(t) = c_1 e^{3x} + c_2 e^{-2x}$.

Now, Let $y_p = A e^{3x} \cdot x$

$y_p' = Ae^{3x} + 3Axe^{3x}$, $y_p'' = 9Axe^{3x} + 6Ae^{3x}$

$\Rightarrow 9Axe^{3x} + 6Ae^{3x} - (Ae^{3x} + 3Axe^{3x}) - 6(Axe^{3x}) = 7e^{3x}$

$\Rightarrow y_p = \frac{7}{5} x e^{3x} \Rightarrow \boxed{y(t) = c_1 e^{3x} + c_2 e^{-2x} + \frac{7}{5} x e^{3x}}$

iii) If $g(t) = A\cos bx + B\sin bx$.

Then we assume $y_p = C\cos bx + D\sin bx$, then substitute y_p in the D.E making sure that y_h and y_p are L.I. Otherwise, we multiply by t^s with $s = 0, 1, 2$.

Example: Solve $y'' - 3y' - 4y = 2\sin t$

sol: $r^2 - 3r - 4 = 0 \iff r_1 = -1,\ r_2 = 4$

$\therefore\ y_h(t) = c_1 e^{-t} + c_2 e^{4t}$.

L.I

Let $y_p = A\sin t + B\cos t$

$y_p' = A\cos t - B\sin t$

$y_p'' = -A\sin t - B\cos t$.

$\Rightarrow (-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t$

$\Rightarrow (-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t$

$\Rightarrow \left.\begin{array}{l} -5A + 3B = 2 \\ -5B - 3A = 0 \end{array}\right\} \Rightarrow \boxed{A = \frac{-5}{17}},\ \boxed{B = \frac{3}{17}}$

$\therefore\ y_p = \frac{-5}{17} \sin t + \frac{3}{17} \cos t$

$\therefore\ y(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{5}{17} \sin t + \frac{3}{17} \cos t .$

Example: $y'' + y = 5 \sin t$

Sol: $r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$

$\therefore\ y_h(t) = c_1 \cos t + c_2 \sin t$

L.I

Let $y_p = (A \sin t + B \cos t) . t$

$\Rightarrow y_p' = (A \sin t + B \cos t) + t(A \cos t - B \sin t)$

$y_p'' = A \cos t - B \sin t + t(-A \sin t - B \cos t) + (A \cos t - B \sin t)$

Substitute in the D.E :

$2A \cos t - 2B \sin t - A t \sin t - B t \cos t = 5 \sin t$

$\Rightarrow \quad -2B = 5 \Rightarrow B = -\frac{5}{2}$

$\therefore\ y_p = -\frac{5}{2} t \cos t ,$

$\therefore$ General solution is

$y(t) = c_1 \cos t + c_2 \sin t - \frac{5}{2} t \cos t .$

iv) If $g(t) = P_n(t)\, e^{\alpha t}$, then we assume

$$y_p = t^s\left(A_0 + A_1 t + \dots + A_n t^n\right) e^{\alpha t}, \quad s = 0, 1, 2$$

v) If $g(t) = P_n(t)\, e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$, then we assume

$$y_p = t^s\Big[\left(A_0 + A_1 t + \dots + A_n t^n\right) e^{\alpha t} \cos \beta t$$

$$+ \left(B_0 + B_1 t + \dots + B_n t^n\right) e^{\alpha t} \sin \beta t\Big]$$

with $s = 0, 1, 2$.

Example: Find the particular Solution for the following:

① $$y'' - 3y' - 4y = -8e^{t} \cos 2t$$

$$y_h(t) = c_1 e^{4t} + c_2 e^{-t}$$

$$y_p = \left[A e^{t} \cos 2t + B e^{t} \sin 2t\right] t^0$$

---

② $$y'' - 3y' - 4y = 3e^{2t} + 2\sin t$$

$$y_h(t) = c_1 e^{4t} + c_2 e^{-t},$$

For y_p we have two Cases:

(i) $y'' - 3y' - 4y = 3e^{2t} \implies y_{p_1} = Ae^{2t}$

(ii) $y'' - 3y' - 4y = 2\sin t \implies y_{p_2} = (B\sin t + C\cdot\cos t)t^{0}$

$\therefore\ y_p = y_{p_1} + y_{p_2} = Ae^{2t} + B\sin t + C\cdot\cos t.$

③ $y'' + y = t + t\sin t$

$y_h(t) = c_1\cos t + c_2\sin t$

We have two subdifferential equations

(i) $y'' + y = t \implies y_{p_1} = (At+B)t^{0}$

(ii) $y'' + y = t\sin t$

$\implies y_{p_2} = [(ct+D)\sin t + (Et+F)\cos t]\,t.$

$\therefore\ y_p = y_{p_1} + y_{p_2} = (At+B) + t(ct+D)\sin t + t(Et+F)\cos t.$

$(At+B) + (ct^2 + Dt)\sin t + (Et^2 + Ft)\cos t$

④ $y'' + 2y' + 2y = t\sin t + 3te^{-t} + 4e^{-t}\sin t.$

$y_h(t) = c_1 e^{-t}\sin t + C_2 e^{-t}\cos t$

Now, we have three subdifferential equations:

(i) $y'' + 2y' + 2y = t\sin t \implies y_{p_1} = [(At+B)\sin t + (ct+D)\cos t]\,t^{0}$

(ii) $y'' + 2y' + 2y = 3te^{-t}$

$$\Rightarrow y_{p_2} = \left[(Et + F)e^{-t}\right]t^0.$$

(iii) $y'' + 2y' + 2y = 4e^{-t}\sin t$

$$\Rightarrow y_{p_3} = \left[Ge^{-t}\sin t + He^{-t}\cos t\right]t^1.$$

$$\therefore y_p = y_{p_1} + y_{p_2} + y_{p_3} = \dots$$

Q13) (page 184) Find General Solution for:

$$y'' + y' + 4y = 2\sinh t.$$

Sol: $r^2 + r + 4 = 0 \Rightarrow r_{1,2} = \frac{-1}{2} \pm \frac{\sqrt{15}}{2}i$

$$\Rightarrow y_h(t) = c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{15}}{2}t\right)$$

Now : $2\sinh t = 2\left(\frac{e^t - e^{-t}}{2}\right) = e^t - e^{-t}$

$\therefore$ We have two sub differentials:

(i) $y'' + y' + 4y = e^t \Rightarrow y_{p_1} = Ae^t$

(ii) $y'' + y' + 4y = e^{-t} \Rightarrow y_{p_2} = Be^{-t}$

$$\therefore y_p = y_{p_1} + y_{p_2} = Ae^t + Be^{-t}.$$

$$y_p' = Ae^t - Be^{-t}, \quad y_p'' = Ae^t + Be^{-t}.$$

$$\Rightarrow Ae^{t}+Be^{-t}+Ae^{t}-Be^{-t}+4Ae^{t}+4Be^{-t}$$
$$= e^{t}-e^{-t}$$

$$\Rightarrow 6Ae^{t}+4Be^{-t}=e^{t}-e^{-t}$$

$$\Rightarrow 6A=1 \Rightarrow \boxed{A=\frac{1}{6}} \quad \& \quad 4B=-1 \Rightarrow \boxed{B=\frac{-1}{4}}$$

Therefore, $y(t)=c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{15}}{2}t\right)+c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{15}}{2}t\right)$
$$+\frac{1}{6}e^{t}-\frac{1}{4}e^{-t}.$$

Q25) (page 184) Determine a suitable form for $y_p(t)$ for $y''-4y'+4y=2t^2+4te^{2t}+t\sin 2t$.

Sol: $y_h(t)=c_1 e^{2t}+c_2 t e^{2t}$.

Now we have three sub differentials:

(i) $y''-4y'+4y=2t^2 \Rightarrow y_{p_1}=At^2+Bt+C$

(ii) $y''-4y'+4y=4te^{2t} \Rightarrow (Dt+E)e^{2t}\cdot\boxed{t^2}=y_{p_2}$

(iii) $y''-4y'+4y=t\sin 2t \Rightarrow$

$\Rightarrow y_{p_3}=(Ft+E)\sin 2t+(Gt+H)\cos 2t$

3.6 Variation of parameters

Consider the 2nd order linear differential equation

$$y'' + p(t)\,y' + q(t)\,y = g(t) \quad \cdots (1)$$

We studied the case when p and q are constants and g(t) is one of the functions: exp, poly, sin, cos. or finite sums and products of these functions.

Question: How can we solve eq.(1) if $g(x)$ is any other function, or p or q is nonconstant ?

Ans: We will use the method of variation of parameters.

Thm 3.6.1: Consider the D.E

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

If p, q and g are continuous on an open Interval I, and if the functions y_1 and y_2 are fundamental set of solutions of the homogeneous

D.E : $y'' + p(t)\,y' + q(t)\,y = 0$, then

the general solution of Eq. (1) is

$$y(t) = y_h + y_p$$

$$= \underbrace{c_1 y_1 + c_2 y_2}_{} + \underbrace{v_1 y_1 + v_2 y_2}_{}, \quad \text{where}$$

$$v_1 = -\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}\,dt, \quad v_2 = \int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}\,dt$$

Example: Find the general solution of the D.E

$$y'' + 4y = 3\csc t.$$

Sol: $y_h(t) = c_1 \cos 2t + c_2 \sin 2t \qquad (r_{1,2} = \pm 2i)$

Let $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$

$$W(y_1,y_2)(t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t$$

$$= \boxed{2} \neq 0$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= v_1 \cos 2t + v_2 \sin 2t.$$

$$V_1 = -\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}\,dt = -\int \frac{\sin 2t\,(3\csc t)}{2}\,dt$$

$$= -\int \frac{\cancel{2}\,\cancel{\sin t}\cos t}{\cancel{2}}\cdot\frac{3}{\cancel{\sin t}}\,dt = -3\int \cos t\,dt$$

$$= -3\sin t.$$

$$V_2 = \int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}\,dt = \int \frac{(\cos 2t)(3\csc t)}{2}\,dt$$

$$= \frac{3}{2}\int \left(1-2\sin^2 t\right)\cdot\frac{1}{\sin t}\,dt = \frac{3}{2}\int(\csc t - 2\sin t)\,dt$$

$$= \frac{3}{2}\ln|\csc t - \cot t| + 3\cos t$$

$$\therefore\ y_p = (-3\sin t)(\cos 2t) + \left[\frac{3}{2}\ln|\csc t - \cot t| + 3\cos t\right](\sin 2t)$$

$\Rightarrow$ The General Solution is

$$y(t) = C_1\cos 2t + C_2\sin 2t + -\underline{3\sin t\cos 2t}^{(*)}$$

$$+\ \frac{3}{2}\sin 2t\ln|\csc t - \cot t| + \underline{3\cos t\sin 2t}^{(**)}.$$

Note: $3\left(\cos t(2\sin t\cos t) - \sin t(2\cos^2 t - 1)\right)$

(*) & (**)

$$= 3\left[2\sin t\cos^2 t - 2\sin t\cos^2 t + \sin t\right] = \boxed{3\sin t}$$

(150)

(Q17)(page 190)

Example: Solve $x^2 y'' - 3xy' + 4y = x^2 \ln x \quad , x > 0$

Sol: $y_h(x)$: $x^2 y'' - 3xy' + 4y = 0$ (Euler Eq.)

Let $t = \ln x$ with $A = 1$, $B = -3$, $C = 4$.

Then $$\frac{d^2y}{dt^2} + (-3-1)\frac{dy}{dt} + 4y = 0$$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$$

Charac. Eq.: $r^2 - 4r + 4 = 0 \iff (r-2)^2 = 0 \iff r_{1,2} = 2$

$$\Rightarrow y_h(t) = c_1 e^{2t} + c_2 t e^{2t}$$

$$= c_1 e^{2\ln x} + c_2 (\ln x) e^{2\ln x}$$

$$= c_1 x^2 + c_2 (\ln x) x^2.$$

$$\Rightarrow y_1 = x^2 \quad \text{and} \quad y_2 = x^2 \ln x$$

Standard form: $y'' - \frac{3}{x} y' + \frac{4}{x^2} y = \ln x \quad , x > 0$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & x + 2x \ln x \end{vmatrix} = x^3 + 2x^3 \ln x - 2x^3 \ln x$$

$$= x^3 \neq 0 \quad , \text{ since } x > 0$$

$$y_p = v_1 y_1 + v_2 y_2 = v_1 x^2 + v_2 x^2 \ln x$$

$$v_1 = -\int \frac{y_2(x)\, g(x)}{W(y_1, y_2)(x)}\, dx = -\int \frac{(x^2 \ln x) \ln x}{x^3}\, dx$$

$$= -\int \frac{(\ln x)^2}{x}\, dx = -\int u^2\, du$$

Let $u = \ln x$
$du = \frac{1}{x}\, dx$

$$= -\frac{u^3}{3} = -\frac{(\ln x)^3}{3}$$

$$v_2 = \int \frac{y_1(x)\, g(x)}{W(y_1, y_2)}\, dx = \int \frac{(x^2) \ln x}{x^3}\, dx = \int \frac{\ln x}{x}\, dx$$

$$= \int u\, du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$$

$$\therefore\ y_p = \frac{-(\ln x)^3}{3} x^2 + \frac{(\ln x)^2}{2} x^2 \ln x$$

$$= \frac{1}{6} x^2 (\ln x)^3$$

$$\therefore\ y(t) = c_1 x^2 + c_2 x^2 \ln x + \frac{1}{6} x^2 (\ln x)^3 .$$

"End of Chapter 3"