Frequency Response ~ Laplace Transform

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$y(t) = h(t) * x(t)$$

$$\xrightarrow{\mathcal{L}} Y(s) = H(s)\,X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} \rightsquigarrow \text{Transfer function}$$

$\Rightarrow$ The frequency response is determined as:

$$H(\omega) = H(s)\Big|_{s=j\omega}$$

EX :- Determine the frequency response of the system with transfer function :-

$$T(s) = \frac{3s+1}{s^2+5}$$

$$H(\omega) = T(s)\Big|_{s=j\omega} = \frac{1+j3\omega}{5-\omega^2}$$

4.8] Steady state response of LTI system using FT "periodic signal."

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}} \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$Y(f) = H(f) X(f)$$

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$$

$$Y(f) = H(f) \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0) = \sum_{n=-\infty}^{\infty} H(nf_0) X_n \delta(f - nf_0)$$

$$y(t) = \sum_{n=-\infty}^{\infty} H(nf_0) X_n e^{j2\pi n f_0 t} = \sum_{n=-\infty}^{\infty} |H(nf_0)| |X_n| e^{j[2\pi n f_0 t + \theta_H(nf_0) + \theta_{X_n}]}$$

SS response

OR

$$y(t) = \sum_{n=-\infty}^{\infty} f_0\, X(nf_0)\, H(nf_0)\, e^{j2\pi n f_0 t}$$

$$y(t) = \sum_{n=-\infty}^{\infty} f_0 |X(nf_0)|\, |H(nf_0)|\, e^{j[2\pi n f_0 t + \theta_x(nf_0) + \theta_H(nf_0)]}$$

EX :- Find the ss response of the system with the frequency response $H(f) = \frac{10}{3+j2\pi f}$ to the signal $x(t) = \sum_{n=-\infty}^{\infty} \operatorname{sinc}(t - nT_0)$

$$y(t) = \sum_{n=-\infty}^{\infty} f_0 X(nf_0) H(nf_0) e^{j2\pi n f_0 t}$$

$$X(f) = F[\operatorname{sinc}(t)] = \Pi(f)$$

$$H(f) = \frac{10}{\sqrt{9+4\pi^2 f^2}} \angle -\tan^{-1}\left(\frac{2\pi f}{3}\right)$$

$$y(t) = \sum_{n=-\infty}^{\infty} \left\{ \frac{10 f_0 \Pi(nf_0)}{\sqrt{9+4\pi^2 n^2 f_0^2}} e^{j\left[2\pi n f_0 t - \tan^{-1}\left(\frac{2\pi n f_0}{3}\right)\right]} \right\}$$

4.9] Hilbert Transform

→ It aims to generate orthogonal signals

The Hilbert transform of $x(t)$ is given by :-

$$x^{H}(t) = x(t) * \frac{1}{\pi t}$$

orthogonal signals

$$F[\operatorname{sgn}(t)] = \frac{1}{j\pi f}$$

$$F\left[\frac{1}{j\pi t}\right] = \operatorname{sgn}(-f)$$

$$= -\operatorname{sgn}(f)$$

$$X^{H}(f) = X(f)\, F\left[\frac{1}{\pi t}\right]$$

$$X^{H}(f) = X(f)\, F\left[\frac{1}{j\pi t}\right] j = -j \operatorname{sgn}(f) X(f)$$

$$\boxed{X^{H}(f) = -j \operatorname{sgn}(f)\, X(f)}$$

Ex :- Find the Hilbert transform of $x(t) = A\cos(\omega_0 t)$

$$X(f) = \frac{A}{2}\delta(f-f_0) + \frac{A}{2}\delta(f+f_0)$$

$$X_H(f) = -j\left[\frac{A}{2}\overbrace{\delta(f-f_0)}^{f>0} + \frac{A}{2}\overbrace{\delta(f+f_0)}^{f<0}\right]\operatorname{sgn}(f)$$

$$= A\left[\frac{\delta(f-f_0) - \delta(f+f_0)}{2j}\right]$$

$$\boxed{x_H(t) = A\sin(\omega_0 t)}$$