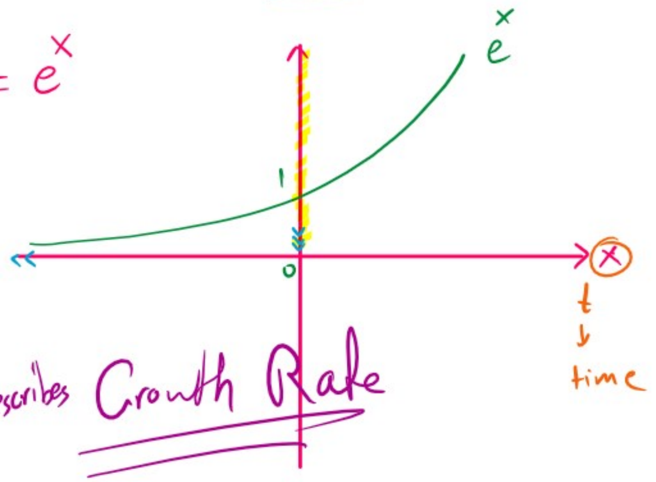
Exponential Change

$y = e^x$



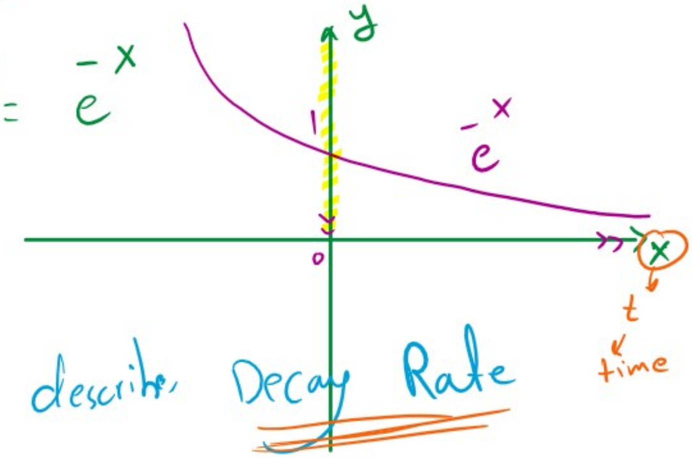
$D = \mathbb{R}$
$R = (0, \infty)$

$\lim_{x \to \infty} e^x = \infty$

$\lim_{x \to -\infty} e^x = 0^+$

e^x describes Growth Rate

$y = e^{-x}$



$D = \mathbb{R}$
$R = (0, \infty)$

$\lim_{x \to \infty} e^{-x} = 0$

$\lim_{x \to -\infty} e^{-x} = \infty$

e^{-x} describes Decay Rate

Differential Equation : is equation with derivative

Differential → derivative مشتقة

Equation معادلة

Exp $y' = 2t$

equation → identify

derivative → rates (changes)

rates (changes) → increases ⇒ growth; decreases ⇒ decay

DE's used to describe natural phenomenas → waves

Population size — Bacteria, virus, animals, human

Radioactivity — اشعاع — فترة نصف الحياة — half-life time

waves

heat ...

decreases $\Rightarrow$ decay

- $P(t)$: population size at time t

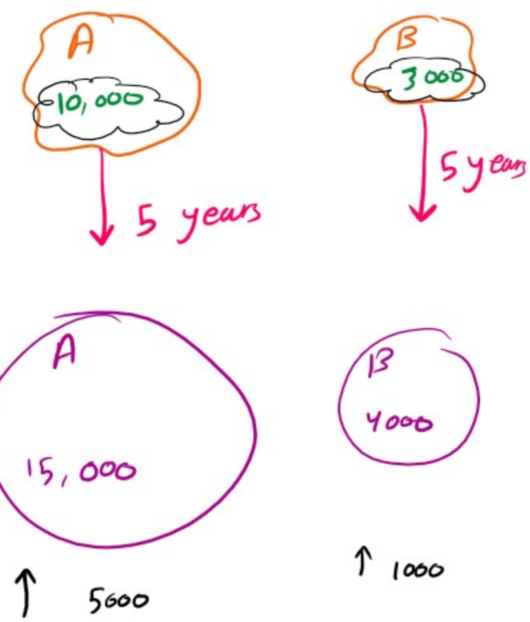
- Assume $p(t)$ increases proportionally to its current size P

$$\frac{dP}{dt} \propto \underline{P}$$

ماذا يعبر؟ التغير في عدد السكان مع الوقت

$$\frac{dP}{dt} = KP \rightarrow \text{DE}$$

اكبر $p(t)$



$K > 0 \Rightarrow$ growth rate

↑ constant

Initial Condition $\boxed{P(0) = P_0}$ عدد السكان في البداية

<u>Initial</u> Condition $\boxed{P(0) = P_0}$ السكان في البداية
I C

عدد

<u>Q.</u> Given DE and IC.
Find population size after 3 years

المطلوب $P(3)$

<u>A.</u> We need to solve $\frac{dP}{dt} = kP$

$$\int \frac{dP}{P} = \int k \, dt$$

P: population
$P > 0$

$$\ln|P| = kt + c$$

$$\ln P = kt + c$$

$$e^{\ln P} = e^{kt+c}$$

$$e^{a_1} e^{a_2} = e^{a_1 + a_2}$$

$$P = e^{c} e^{kt}$$

$$D = e^{c}$$

$$\boxed{P(t) = D e^{kt}}, \quad P(0) = P_0$$

$$\boxed{P(t) = D\,e} \quad ,$$

$$P(0) = D\,e^{K(0)}$$

$$P_0 = D(1) \Rightarrow D = P_0$$

$$\boxed{P(t) = P_0\, e^{Kt}} \rightarrow \text{solution of } \frac{dP}{dt} = KP$$

$$P(3) = P_0\, e^{3K}$$

If $K > 0 \Rightarrow$ growth

$K < 0 \Rightarrow$ decay

Exp (Growth of Bacteria)

30 A colony of bacteria grows with time ... ($K > 0$)

- At the end of 3 hours there are 10,000 bacteria
- At the end of 5 hours there are 40,000 bacteria

How many bacteria were present initially?

المطلوب $\rightarrow P_0$

$$P(t) = P_0\, e^{Kt} \quad , \quad K > 0$$

$$P(3) = \boxed{P_0\, e^{K(3)} = 10,000} \text{ — (1)}$$

$$\boxed{\begin{aligned} p(3) &= P_0\, e^{k(3)} = 10{,}000 \;\text{—}\; ① \\ p(5) &= P_0\, e^{k(5)} = 40{,}000 \;\text{—}\; ② \end{aligned}}$$

$$\frac{②}{①} \Rightarrow \frac{40{,}000}{10{,}000} = \frac{\cancel{P_0}\, e^{k5}}{\cancel{P_0}\, e^{k3}}$$

$$4 = e^{5k-3k}$$

$e \approx 2.718$

$$4 = e^{2k} \Rightarrow \ln 4 = \ln e^{2k}$$

$$\ln 2^2 = 2k\,(\ln e) \quad \ln e \to 1$$

$$\frac{2\ln 2}{2} = \frac{\cancel{2}k}{\cancel{2}}$$

$$\boxed{k = \ln 2}$$

$$① \Rightarrow 10{,}000 = P_0\, e^{3k}$$

$$P_0 = \frac{10{,}000}{e^{3\ln 2}} = \frac{10{,}000}{e^{\ln 8}} = \frac{10{,}000}{8} = 1250$$

Exp 3 (Pop. Decay)

$y(t)$: # of infected people by disease at time t (years)

y at time t (years)

Assume # of people cured is proportional to the # y

مصاب

نقص 20%

Suppose in one year, the number y is reduced by 20% ??

If 10,000 case today ⟹ how many years will it take to reduce the number y to 1000 case?

المطلوب Find time t^* such that $y(t^*) = 1000$

$$y(t) = y_0 e^{kt}$$

$k < 0$

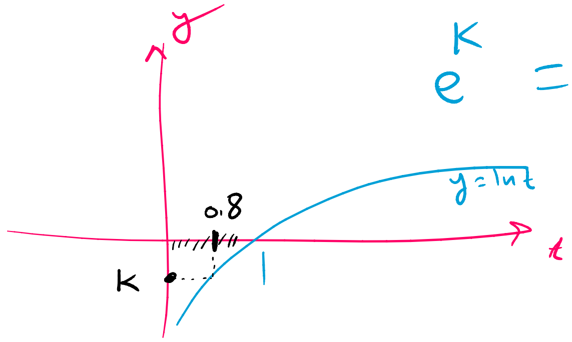
$$\boxed{y(t) = 10000\, e^{kt}}$$

$$y(1) = \frac{80}{100} y_0 = \frac{80}{\cancel{100}} (10\cancel{000}) = 8\,000$$

$$y(1) = 10\cancel{000}\, e^{k(1)} = 8\cancel{000}$$

$$10\, e^{k} = 8$$

$$\Rightarrow \ln e^{k} = \ln 0.8$$



$$e^{K} = \frac{8}{10} \Rightarrow \ln e^{K} = \ln 0.8$$

رقم سالب $\leftarrow$ $\boxed{K = \ln 0.8}$ < 0

Decay

$$y = y_0 e^{kt}$$

$$y(t^*) = y_0 e^{kt^*} = 1000$$

$$10,\cancel{000}\, e^{\ln(0.8)\, t^*} = 1\cancel{000}$$

$$e^{t^* \ln(0.8)} = \frac{1}{10}$$

$$t^* \ln(0.8) = \ln 0.1$$

$$t^* = \frac{\ln(0.1)}{\ln(0.8)} = \frac{\text{سالب}}{\text{سالب}} > 0$$

time

$$t^* \approx 10{,}32 \text{ years.}$$

Radioactivity إشعاعي

Decay $\boxed{k<0}$

8gm مدة نصف كمية 4gm

τ time

$$Q(t) = Q_0 e^{-kt} \quad , \quad k>0$$

Q_0: initial quantity 8gm

$Q(t)$: quantity of material available at time

متوفر

Half-life time τ is time such that

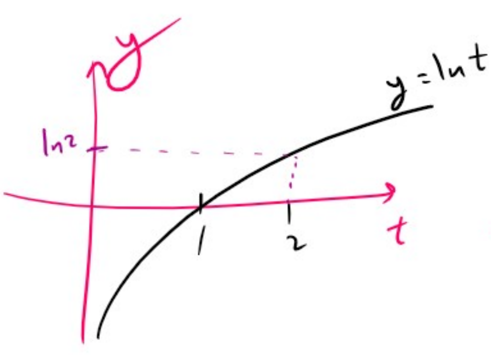
$$Q(\tau) = \frac{1}{2} Q_0$$

$$\cancel{Q_0} e^{-k\tau} = \frac{1}{2} \cancel{Q_0}$$

$$e^{-k\tau} = \frac{1}{2} \Rightarrow \ln e^{-k\tau} = \ln \frac{1}{2}$$

$$-k\tau (\ln e) = \ln 1 - \ln 2$$

$$-k\tau = 0 - \ln 2$$



$$\boxed{\tau = \frac{\ln 2}{k}}$$

half-life time

موجب

موجبة

موجب موجبه

Exp⁴ Carbon 14 has half-life time 5700 years. Find age of a sample in which 10% of the radioactive material has decayed.

اختفى

Find time t^* such that $Q(t^*) = \frac{90}{100} Q_0$

$$Q(t) = Q_0 e^{-kt} \quad , \; k > 0$$

$$Q(t^*) = Q_0 e^{-kt^*}$$

$$\frac{90}{100} Q_0 = Q_0 e^{-kt^*}$$

$$0.9 = e^{-kt^*}$$

$$\ln 0.9 = -k t^* \ln e \qquad (\ln e = 1)$$

$$\boxed{t^* = \frac{\ln 0.9}{-k}}$$

سالب

موجب

موجب

??

$$= \frac{\ln 0.9}{-\frac{\ln 2}{5700}} \approx 866 \text{ years}$$

$$\tau = \frac{\ln 2}{k}$$

$$5700 = \frac{\ln 2}{k}$$

$$\boxed{k = \frac{\ln 2}{5700}} > 0$$

موجب

رجب ←

5700

Exp The half-life time of a radioactive material is $\ln 8$ years.
If 10 gm of this material is released into atmosphere. How many years will it take for 80% of the material to decay?

$$T = \frac{\ln 2}{k} \Rightarrow \ln 8 = \frac{\ln 2}{k} \Rightarrow k = \frac{\ln 2}{\ln 8}$$

$$k = \frac{\ln 2}{\ln 2^3} = \frac{\ln 2}{3\ln 2} = \frac{1}{3}$$

Find t^* s.t

$Q(t^*) = \frac{20}{100} Q_0$

$$Q(t) = Q_0 e^{-kt}$$

available

$$Q(t^*) = Q_0 e^{-kt^*}$$

$$\frac{20}{100} \cancel{Q_0} = \cancel{Q_0} e^{-kt^*}$$

$$0.2 = e^{-kt^*} \Rightarrow \ln 0.2 = -kt^*$$

$$\Rightarrow t^* = \frac{\ln 0.2}{-k} = \frac{\ln 0.2}{-\frac{1}{3}} = -3(\ln 0.2)$$

سالب

$-K \qquad -\frac{1}{3}$

$$t^* = -3\ln\frac{2}{10} = -3\ln\frac{1}{5} = -3\,(-\ln 5)$$

$$\boxed{t^* = 3\ln 5}$$

Exp Assume population of mice doubles in 2 years. How many years will it take this population to be tribble?

$P(t)$: Pop. size at time t $\qquad$ P_0: intial size

$$p(2) = 2P_0$$

Find time t^* such that $P(t^*) = 3P_0$

$$P(t) = P_0 e^{kt}$$

$K > 0 \rightarrow$ doubles growth

$$p(2) = P_0 e^{k(2)}$$

$$2\cancel{P_0} = \cancel{P_0} e^{K(2)}$$

$$2 = e^{2K} \Rightarrow \ln 2 = 2K \Rightarrow \boxed{K = \frac{\ln 2}{2}}$$

$2 = e \Rightarrow \ln 2$ ...

$$P(t) = P_0 e^{kt}$$

$$P(t^*) = P_0 e^{kt^*}$$

$$3\cancel{P_0} = \cancel{P_0} e^{kt^*}$$

$$3 = e^{kt^*} \Rightarrow \ln 3 = kt^* \Rightarrow t^* = \frac{\ln 3}{k}$$

$$t^* = \frac{\ln 3}{\frac{\ln 2}{2}} = \frac{2}{\ln 2} \cdot \ln 3 = \frac{2\ln 3}{\ln 2} = \frac{\ln 3^2}{\ln 2}$$

$$t^* = \frac{\ln 9}{\ln 2}$$ [illegible]