3.3 Linear Independence

Definition The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space V are said to be **linearly independent** if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

implies that all the scalars $c_1, \ldots, c_n$ must equal 0.

EXAMPLE 1 The vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are linearly independent, since if

$$c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 2c_2 &= 0 \end{aligned}$$

and the only solution to this system is $c_1 = 0, c_2 = 0$.

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space V are said to be **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

EXAMPLE 2 Let $\mathbf{x} = (1, 2, 3)^T$. The vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and $\mathbf{x}$ are linearly dependent, since

$$\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 - \mathbf{x} = \mathbf{0}$$

(In this case $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = -1$.) ■

If there are nontrivial choices of scalars for which the linear combination $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ equals the zero vector, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent. If the *only* way the linear combination $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ can equal the zero vector is for all the scalars $c_1, \ldots, c_n$ to be 0, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

EXAMPLE 3 Which of the following collections of vectors are linearly independent in $\mathbb{R}^3$?

(a) $(1, 1, 1)^T, (1, 1, 0)^T, (1, 0, 0)^T$

(b) $(1, 0, 1)^T, (0, 1, 0)^T$

Solution

(a) These three vectors are linearly independent. To verify this, we must show that the only way for

$$c_1(1, 1, 1)^T + c_2(1, 1, 0)^T + c_3(1, 0, 0)^T = (0, 0, 0)^T \tag{4}$$

is if the scalars c_1, c_2, c_3 are all zero. Equation (4) can be written as a linear system with unknowns c_1, c_2, c_3:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + c_2 \phantom{{}+c_3} &= 0 \\ c_1 \phantom{{}+c_2+c_3} &= 0 \end{aligned}$$

The only solution of this system is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

(b) If

$$c_1(1, 0, 1)^T + c_2(0, 1, 0)^T = (0, 0, 0)^T$$

then

$$(c_1, c_2, c_1)^T = (0, 0, 0)^T$$

so $c_1 = c_2 = 0$. Therefore, the two vectors are linearly independent.

Geometric Interpretation

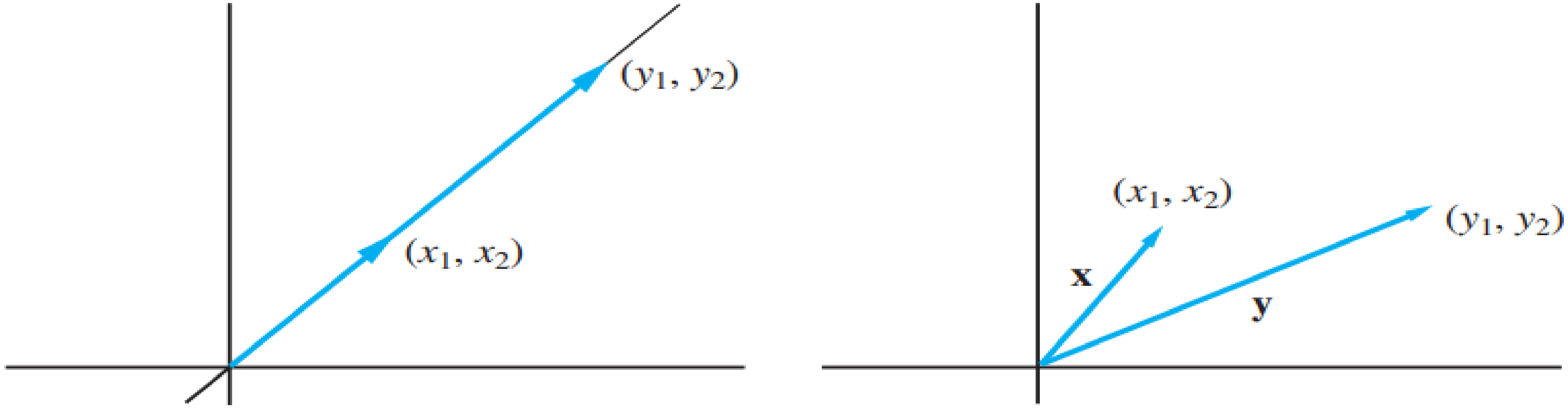
If **x** and **y** are linearly dependent in $\mathbb{R}^2$, then

$$c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0}$$

where c_1 and c_2 are not both 0. If, say, $c_1 \neq 0$, we can write

$$\mathbf{x} = -\frac{c_2}{c_1}\mathbf{y}$$

If two vectors in $\mathbb{R}^2$ are linearly dependent, one of the vectors can be written as a scalar multiple of the other. Thus, if both vectors are placed at the origin, they will lie along the same line (see Figure 3.3.1).



(a) **x** and **y** linearly dependent

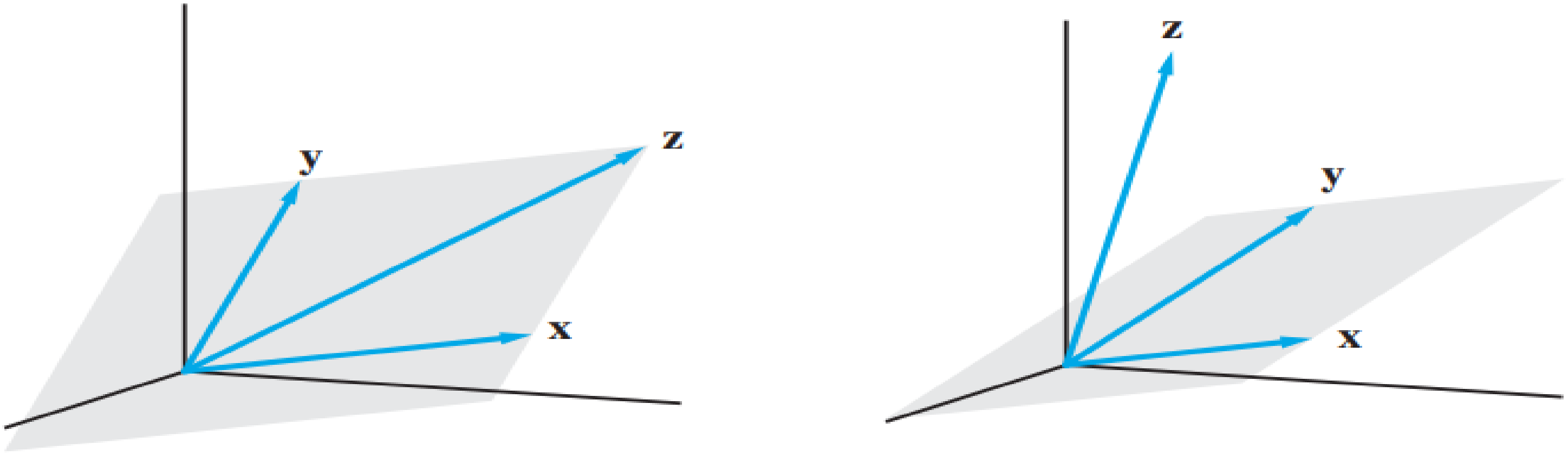
(b) **x** and **y** linearly independent

Figure 3.3.1.

If

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

are linearly independent in $\mathbb{R}^3$, then the two points (x_1, x_2, x_3) and (y_1, y_2, y_3) will not lie on the same line through the origin in 3-space. Since $(0, 0, 0)$, (x_1, x_2, x_3), and (y_1, y_2, y_3) are not collinear, they determine a plane. If (z_1, z_2, z_3) lies on this plane, the vector $\mathbf{z} = (z_1, z_2, z_3)^T$ can be written as a linear combination of $\mathbf{x}$ and $\mathbf{y}$, and hence $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly dependent. If (z_1, z_2, z_3) does not lie on the plane, the three vectors will be linearly independent (see Figure 3.3.2).



(a) (b)

Figure 3.3.2.

Theorem 3.3.1 *Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n vectors in $\mathbb{R}^n$ and let $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$. The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ will be linearly dependent if and only if X is singular.*

Proof The equation

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$$

can be rewritten as a matrix equation

$$X\mathbf{c} = \mathbf{0}$$

This equation will have a nontrivial solution if and only if X is singular. Thus, $\mathbf{x}_1, \ldots, \mathbf{x}_n$ will be linearly dependent if and only if X is singular. ■

Remark:

We can use Theorem 3.3.1 to test whether n vectors are linearly independent in $\mathbb{R}^n$. Simply form a matrix X whose columns are the vectors being tested. To determine whether X is singular, calculate the value of $\det(X)$. If $\det(X) = 0$, the vectors are linearly dependent. If $\det(X) \neq 0$, the vectors are linearly independent.

EXAMPLE 4 Determine whether the vectors $(4, 2, 3)^T$, $(2, 3, 1)^T$, and $(2, -5, 3)^T$ are linearly dependent.

Solution

Since

$$\begin{vmatrix} 4 & 2 & 2 \\ 2 & 3 & -5 \\ 3 & 1 & 3 \end{vmatrix} = 0$$

the vectors are linearly dependent. ■

Remark:

To determine whether k vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k$ in $\mathbb{R}^n$ are linearly independent we can rewrite the equation

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$$

as a linear system $X\mathbf{c} = \mathbf{0}$, where $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k)$. If $k \neq n$, then the matrix X is not square, so we cannot use determinants to decide whether the vectors are linearly independent. The system is homogeneous, so it has the trivial solution $\mathbf{c} = \mathbf{0}$. It will have nontrivial solutions if and only if the row echelon forms of X involve free variables. If there are nontrivial solutions, then the vectors are linearly dependent. If there are no free variables, then $\mathbf{c} = \mathbf{0}$ is the only solution, and hence the vectors must be linearly independent.

EXAMPLE 5 Given

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 7 \\ 7 \end{pmatrix}$$

To determine whether the vectors are linearly independent, we reduce the system $X\mathbf{c} = \mathbf{0}$ to row echelon form:

$$\left(\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 0 \\ 2 & 1 & 7 & 0 \\ 3 & -2 & 7 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since the echelon form involves a free variable c_3, there are nontrivial solutions and hence the vectors must be linearly dependent. ■

EXAMPLE 6 To test whether the vectors

$$p_1(x) = x^2 - 2x + 3, \quad p_2(x) = 2x^2 + x + 8, \quad p_3(x) = x^2 + 8x + 7$$

are linearly independent, set

$$c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = 0x^2 + 0x + 0$$

Grouping terms by powers of x, we get

$$(c_1 + 2c_2 + c_3)x^2 + (-2c_1 + c_2 + 8c_3)x + (3c_1 + 8c_2 + 7c_3) = 0x^2 + 0x + 0$$

Equating coefficients leads to the system

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ -2c_1 + c_2 + 8c_3 &= 0 \\ 3c_1 + 8c_2 + 7c_3 &= 0 \end{aligned}$$

The coefficient matrix for this system is singular and hence there are nontrivial solutions. Therefore, p_1, p_2, and p_3 are linearly dependent.

Theorem 3.3.2 *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *be vectors in a vector space* V. *A vector* $\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ *can be written uniquely as a linear combination of* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *if and only if* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are linearly independent.*

Proof If $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, then $\mathbf{v}$ can be written as a linear combination

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n \tag{5}$$

Suppose that $\mathbf{v}$ can also be expressed as a linear combination

$$\mathbf{v} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_n\mathbf{v}_n \tag{6}$$

We will show that, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, then $\beta_i = \alpha_i$, $i = 1, \ldots, n$, and if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent, then it is possible to choose the β_i's different from the α_i's.

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, then subtracting (6) from (5) yields

$$(\alpha_1 - \beta_1)\mathbf{v}_1 + (\alpha_2 - \beta_2)\mathbf{v}_2 + \cdots + (\alpha_n - \beta_n)\mathbf{v}_n = \mathbf{0} \tag{7}$$

By the linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the coefficients of (7) must all be 0. Hence

$$\alpha_1 = \beta_1,\ \alpha_2 = \beta_2,\ \ldots,\ \alpha_n = \beta_n$$

Thus, the representation (5) is unique when $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

On the other hand, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent, then there exist $c_1, \ldots, c_n$, not all 0, such that

$$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \tag{8}$$

Now if we set

$$\beta_1 = \alpha_1 + c_1,\ \beta_2 = \alpha_2 + c_2,\ \ldots,\ \beta_n = \alpha_n + c_n$$

then, adding (5) and (8), we get

$$\begin{aligned}\mathbf{v} &= (\alpha_1 + c_1)\mathbf{v}_1 + (\alpha_2 + c_2)\mathbf{v}_2 + \cdots + (\alpha_n + c_n)\mathbf{v}_n \\ &= \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_n\mathbf{v}_n\end{aligned}$$

Since the c_i's are not all 0, $\beta_i \neq \alpha_i$ for at least one value of i. Thus, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent, the representation of a vector as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is not unique. ■

The Vector Space $C^{(n-1)}[a, b]$

Definition

Let $f_1, f_2, \ldots, f_n$ be functions in $C^{(n-1)}[a, b]$, and define the function $W[f_1, f_2, \ldots, f_n](x)$ on $[a, b]$ by

$$W[f_1, f_2, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

The function $W[f_1, f_2, \ldots, f_n]$ is called the **Wronskian** of $f_1, f_2, \ldots, f_n$.

Theorem 3.3.3 *Let $f_1, f_2, \ldots, f_n$ be elements of $C^{(n-1)}[a,b]$. If there exists a point x_0 in $[a,b]$ such that $W[f_1, f_2, \ldots, f_n](x_0) \neq 0$, then $f_1, f_2, \ldots, f_n$ are linearly independent.*

Proof If $f_1, f_2, \ldots, f_n$ were linearly dependent, then there exist scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \tag{10}$$

for each x in $[a,b]$. Taking the derivative with respect to x of both sides of (10) yields

$$c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) = 0$$

If we continue taking derivatives of both sides, we end up with the system

$$\begin{aligned}
c_1 f_1(x) \quad &+ \quad c_2 f_2(x) \quad + \cdots + \quad c_n f_n(x) \quad = 0 \\
c_1 f_1'(x) \quad &+ \quad c_2 f_2'(x) \quad + \cdots + \quad c_n f_n'(x) \quad = 0 \\
&\vdots \\
c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) &+ \cdots + c_n f_n^{(n-1)}(x) = 0
\end{aligned}$$

For each fixed x in $[a, b]$, the matrix equation

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{11}$$

will have the same nontrivial solution $(c_1, c_2, \ldots, c_n)^T$. Thus, if $f_1, \ldots, f_n$ are linearly dependent in $C^{(n-1)}[a, b]$, then, for each fixed x in $[a, b]$, the coefficient matrix of system (11) is singular. If the matrix is singular, its determinant is zero.

EXAMPLE 7 Show that e^x and e^{-x} are linearly independent in $C(-\infty, \infty)$.

Solution

$$W[e^x, e^{-x}] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Since $W[e^x, e^{-x}]$ is not identically zero, e^x and e^{-x} are linearly independent. ■

EXAMPLE 9 Show that the vectors $1, x, x^2$, and x^3 are linearly independent in $C((-\infty, \infty))$.

Solution

$$W[1, x, x^2, x^3] = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12$$

Since $W[1, x, x^2, x^3] \not\equiv 0$, the vectors are linearly independent. ■

EXAMPLE 8 Consider the functions x^2 and $x|x|$ in $C[-1, 1]$. Both functions are in the subspace $C^1[-1, 1]$ (see Example 7 of Section 3.2), so we can compute the Wronskian

$$W[x^2, x|x|] = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} \equiv 0$$

Since the Wronskian is identically zero, it gives no information as to whether the functions are linearly independent. To answer the question, suppose that

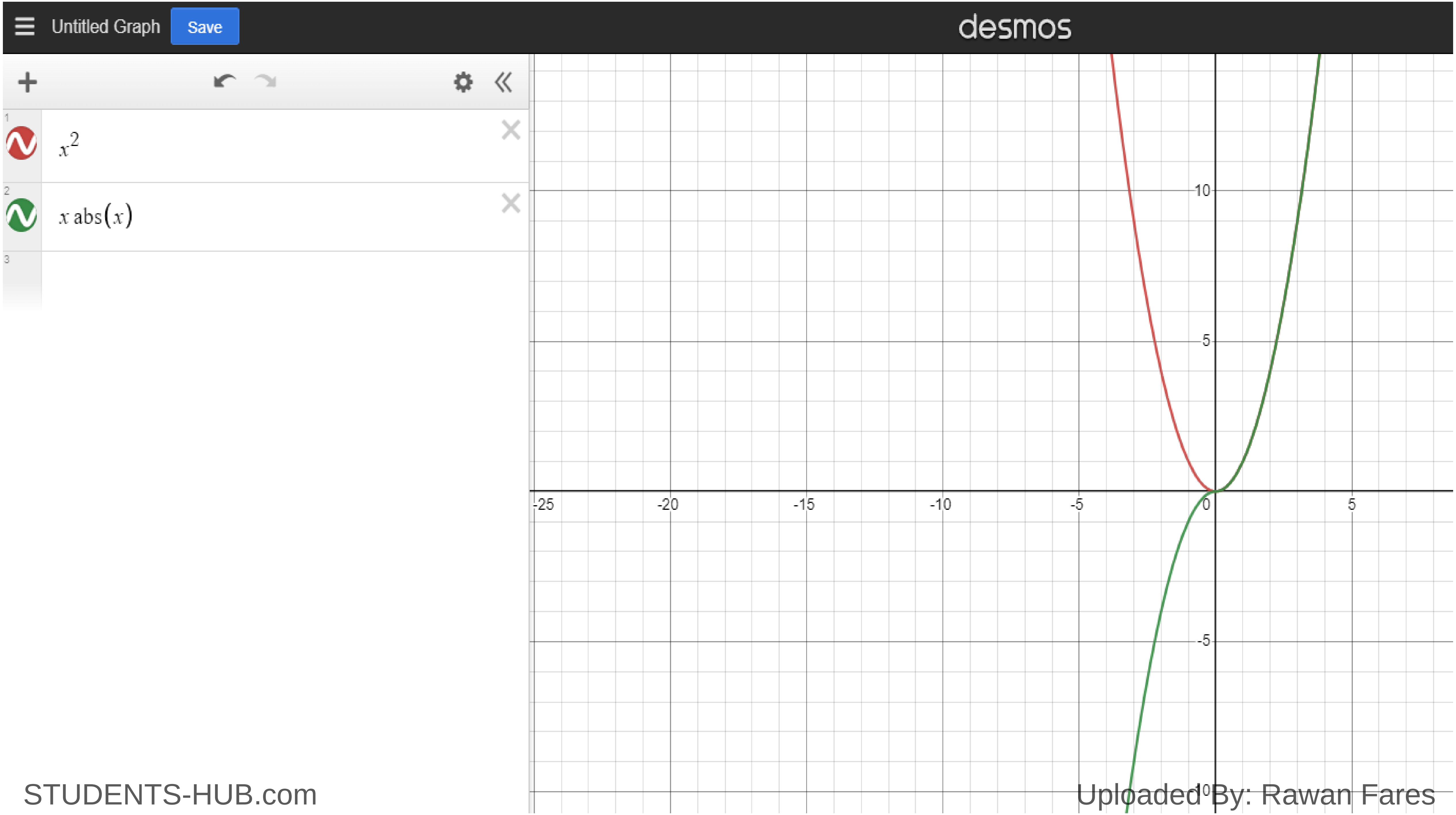
$$c_1x^2 + c_2x|x| = 0$$

for all x in $[-1, 1]$. Then, in particular for $x = 1$ and $x = -1$, we have

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0$$

and the only solution of this system is $c_1 = c_2 = 0$. Thus, the functions x^2 and $x|x|$ are linearly independent in $C[-1, 1]$ even though $W[x^2, x|x|] \equiv 0$.

This example shows that the converse of Theorem 3.3.3 is not valid. ■


Untitled Graph
Save
desmos
1
x^2
2
$x \operatorname{abs}(x)$
3
10
5
-5
-25
-20
-15
-10
-5
0
5

SECTION 3.3 EXERCISES

4. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times 2}$:

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

8. Determine whether the following vectors are linearly independent in P_3:

(a) $1, x^2, x^2 - 2$

(b) $2, x^2, x, 2x + 3$

(c) $x + 2, x + 1, x^2 - 1$

(d) $x + 2, x^2 - 1$

9. For each of the following, show that the given vectors are linearly independent in $C[0, 1]$:

(a) $\cos \pi x, \ \sin \pi x$

(b) $x^{3/2}, \ x^{5/2}$

(c) $1, \ e^x + e^{-x}, \ e^x - e^{-x}$

(d) $e^x, \ e^{-x}, \ e^{2x}$

19. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for the vector space V, and let $\mathbf{v}$ be any other vector in V. Show that $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.

20. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly independent vectors in a vector space V. Show that $\mathbf{v}_2, \ldots, \mathbf{v}_n$ cannot span V.