Exercises :

3.3.0 : True or False :

a. If f is continuous on $[a,b]$ and $J := f([a,b])$, then J is a closed bounded interval ? True .

proof : By Extreme value Thm

$\exists\, x_M, x_m \in [a,b]$ s.t $f(x_m) = \alpha := \inf\{f(x) : x \in [a,b]\}$

and $f(x_M) = \beta := \sup\{f(x) : x \in [a,b]\}$

Thus, $\alpha, \beta \in J$.

If $t \in (\alpha, \beta)$ then By Intermediate value Thm

there is a $x \in [a,b]$ s.t $f(x) = t$ $\rightsquigarrow$ i.e $t \in J$.

we have shown that $[\alpha, \beta] \subseteq J$.

on the other hand, if $t \in J$ then $t = f(x)$ for some $x \in [a,b]$

so By the choice of α and β, $\alpha \leq t \leq \beta$ $\rightsquigarrow$ i.e $J \subseteq [\alpha, \beta]$.

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b. If f and g are continuous on $[a,b]$, if $f(a) < g(a)$ and $f(b) > g(b)$ then there is a $c \in [a,b]$ s.t $f(c) = g(c)$. True .

let $h(x) = g(x) - f(x)$ By hypothesis $h(a) > 0$ and $h(b) < 0$.

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Hence, By Intermediate value Thm :

there is a $c \in [a,b]$ s.t $h(c) = 0$

$$g(c) - f(c) = 0$$

$$\text{i.e} \quad g(c) = f(c)$$

c. suppose f and g are defined and finite valued on an open interval I which contains a, that f is continuous at a, and $f(a) \neq 0$, Then g is cont. at a iff fg is continuous at a. True

If f is continuous at a then fg is continuous Too. (By Thm 2)

Conversely, if fg is continuous at a and f is continuous and $f(a) \neq 0$ Then $g = \frac{fg}{f}$ is continuous at a (By Thm 2).

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d. suppose that f and g are defined and finite valued on $\mathbb{R}$. If f and $g \circ f$ are continuous on $\mathbb{R}$, then g is continuous on $\mathbb{R}$. False

let
$$\left.\begin{aligned} f(x) &= 2 - x && \text{for } x \leq 1 \\ f(x) &= \frac{1}{x} && \text{for } x \geq 1 \end{aligned}\right\} \text{Cont.}$$

$$\left.\begin{aligned} g(x) &= 1 - x && \text{for } x \leq 1 \\ g(x) &= -x && \text{for } x > 1 \end{aligned}\right\} \rightarrow \text{discontinuous}$$

since $f(x) > 0 \ \forall x$ and g continuous on $(0, \infty)$ it is clear that f and $g \circ f$ are continuous But g is Not.

3.3.1: use limit Thm's to show that the following functions are conti. on $[0,1]$.

a. $f(x) = \dfrac{e^{x^2}\sqrt{\sin x}}{\cos x}$

e^{x^2} and $\sqrt{\sin x}$ are continuous on $\mathbb{R}$.

and since $\cos x \neq 0$ on $[0,1]$ Then By Thm 2 $\left(\frac{f}{g}\right)$.

$\dfrac{e^{x^2}\sqrt{\sin x}}{\cos x}$ is continuous on $\mathbb{R}$.

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b.
$$f(x) = \begin{cases} \dfrac{x^2+x-2}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$$

→ $\dfrac{x^2+x-2}{x-1}$ is continuous on $[0,1)$ since $x-1 \neq 0$ on $[0,1)$.

→ since $f(x) \to 3$ as $x \to 1$ and $f(1) = 3$ By RMK 1.

So $f(x)$ is continuous on $[0,1]$.

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c.
$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$x \neq 0$ for $x \in (0,1]$, So $f(x) = e^{-\frac{1}{x}}$ is continuous on $(0,1]$ By Thm 2.

since $-\dfrac{1}{x} \to -\infty$ as $x \to 0^+$ implies $e^{-\frac{1}{x}} \to e^{-\infty} = 0 = f(0)$.

it follows that from RMK1 that f is continuous on $[0,1]$.

d. $f(x) = \begin{cases} \sqrt{x}\,\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\sqrt{x}$ continuous and $\sin\frac{1}{x}$ continuous so $\sqrt{x}\sin\frac{1}{x}$ continuous on $x>0$ By Thm 2.

since $0 \leq \frac{\sqrt{x}}{\sin\frac{1}{x}} \leq \sqrt{x}$ it follows By squeez Thm that

$f(x) \to 0 := f(0)$ as $x \to 0^+$

Thus, f is continuous on $[0,1]$

3.3.2 : prove that there is at least one $x \in \mathbb{R}$ which satisfies the following equation.

a. $e^x = x^3$.

consider $f(x) = e^x - x^3$ $\rightarrow$ $f(x)$ is continuous and

$f(-1) = \frac{1}{e} + 1$

?

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cont. on $[a,b]$
$f(a) \cdot f(b) < 0$

b. $e^x = 2\cos x + 1$

$f(x) = e^x - 2\cos x - 1 \rightsquigarrow f(x)$ is continuous

$f(0) = 1 - 2 - 1 = -2 < 0 < e + 1 = f(1)$

Hence, By intermediate value Theorem,

there is an x (between $0, 1$) s.t $f(x) = 0$.

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c. $2^x = 2 - 3x$.

$f(x) = 2^x - 2 + 3x \longrightarrow f(x)$ is continuous.

$f(0) = -1 < 0$

$f(1) = 3 > 0$

Hence, By Intermediate value Thm.

ther is an x (between 0 and 1) s.t $f(x) = 0$.

2.3.3: If $f: [a,b] \to \mathbb{R}$ is continuous, prove that $\sup_{x\in[a,b]} |f(x)|$ is finite

$|f|$ is continuous on $[a,b]$ ✓

Hence, it follows from the Extreme value Thm that:

$|f|$ is bounded on $[a,b]$ i.e $\sup_{x\in[a,b]} |f(x)|$ is finite

3.3.4: If $f : [a,b] \to [a,b]$ is continuous, then f has a fixed point; that is there is a $c \in [a,b]$ s.t $f(c) = c$.

let $g(x) = f(x) - x$.

since $f(x) \in [a,b]$ for all $x \in [a,b]$ it is clear that

$f(a) \geq a$ and $f(b) \leq b$.

Therefor, $g(a) = f(a) - a \geq 0$

and $g(b) = g(b) - b \leq 0$

since g is continuous on $[a,b]$, it follows from the Intermediate value Thm

that there is a $c \in [a,b]$ s.t $g(c) = 0$

$$g(c) = f(c) - c$$
$$0 = f(c) - c$$
$$f(c) = c$$

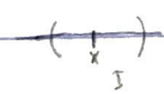
335: If f is a real function which is continuous at $a \in \mathbb{R}$ and if $f(a) < M$ for some $M \in \mathbb{R}$, prove that there is an open interval I containing a s.t $f(x) < M$ $\forall x \in I$.

since $M - f(x) > 0$,

?!

$g(x) > 0$

$\exists$ an interval I



$g(x_0) > 0$

3.5.6 : Show that there exist nowhere continuous functions f and g whose sum $f+g$ is continuous on $\mathbb{R}$. Show that the same is true for the product of functions.

let
$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$$

and $g(x) = 1 - f(x)$ Then $f(x) + g(x) = 1$
and $f(x)\,g(x) = 0$, $\forall x \in \mathbb{R}$.

Hence, $f+g$ and fg are continuous on $\mathbb{R}$ even though f and g are nowhere continuous.

3.3.7 : suppose that $a \in \mathbb{R}$, that I is open interval containing a, that $f, g : I \to \mathbb{R}$, and that f is continuous at a. prove that g is continuous at a iff $f+g$ is continuous at a.

$\rightarrow$ If g is continuous at a Then By (Thm 2)
$f+g$ is continuous

$\leftarrow$ if f and $f+g$ are continuous at a then $g = (f+g) - f$
is continuous.

3.3.8: suppose that $f: \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x) + f(y)$ for each $x, y \in \mathbb{R}$.

a. show that $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

By induction:

$\to$ $f(0) = f(0+0) = f(0) + f(0) = 2f(0) \rightsquigarrow f(0) = 0$.

$\to$ For each $x \in \mathbb{R}$, $f(2x) = f(x+x) = f(x) + f(x) = 2f(x)$.

$\to$ And $0 = f(0) = f(x-x) = f(x) + f(-x)$ implies $f(-x) = -f(x)$.

$\rightsquigarrow$ So $f(nx) = f(x + \dots + x) = nf(x)$ holds for all $n \in \mathbb{Z}$.

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b. prove that $f(qx) = qf(x)$ for all $x \in \mathbb{R}$ and $q \in \mathbb{Q}$.

By part a,

$f(x) = f\left(\frac{mx}{m}\right) = m f\left(\frac{x}{m}\right)$

Thus $f\left(\frac{x}{m}\right) = \frac{f(x)}{m}$ for $m \in \mathbb{N}$.

If $q \in \mathbb{Q}$ then $q = \frac{n}{m}$ so $f\left(\frac{nx}{m}\right) = n f\left(\frac{x}{m}\right)$

$= \frac{n}{m} f(x)$

$= q f(x)$

for $x \in \mathbb{R}$.

c. prove that f is continuous at 0 iff f is continuous on $\mathbb{R}$.

$\rightarrow$ suppose f is continuous at 0 and $x \in \mathbb{R}$.

If $x_n \to x$ then $x_n - x \to 0$

i.e $|f(x_n) - f(x)| = |f(x_n - x)| \to 0$ as $n \to \infty$.

Thus, f is continuous on $\mathbb{R}$.

$\leftarrow$ trivial.

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d. prove that f is continuous at 0, then there is an $m \in \mathbb{R}$ s.t $f(x) = mx$ for all $x \in \mathbb{R}$.

ثبت

let $m = f(1)$ and fix $x \in \mathbb{R}$.

Choose $q_n \in \mathbb{Q}$ s.t $q_n \to x$ as $n \to \infty$. Then By b and c

$$f(x) = f\left(\lim_{n\to\infty} q_n \cdot 1\right) = \lim_{n\to\infty} f(q_n \cdot 1) = \lim_{n\to\infty} q_n f(1) = mx.$$

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