Homework 5. Reem Ali.

Abistract 1 (chapter 6)

2. Find Aut( Z ) :

let $\phi: Z \to Z$ be an Automorphism

$\phi(n) = n\,\phi(1)$

$\rightsquigarrow$ 1 is generator of $Z$ so $\phi(1)$ is a generator of Aut( Z ).

$\rightsquigarrow \phi(1) = \pm 1 \rightsquigarrow$ If $\phi(1) = 1$ then $\phi(n) = n$

$\phi(1) = -1$ then $\phi(n) = -n$.

So automorphisim of $Z$ has two element $\phi(n) = n$

$\phi(n) = -n$.

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4. show that $U(8)$ is not isomorphism to $U(10)$ :

$U(8) = \{1,3,5,7\}$ , $U(10) = \{1,3,7,9\}$

since $U(8)$ is Not cycle But $U(10)$ is cyclic

we conclude $U(8) \not\approx U(10)$

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5. show $U(8)$ is issmorphic to $U(12)$ :

$U(8) = \{1,3,5,7\}$ , $U(12) = \{1,5,7,11\}$

let $\phi$:

$1 \longrightarrow 1$

$3 \longrightarrow 5$

$5 \longrightarrow 7$

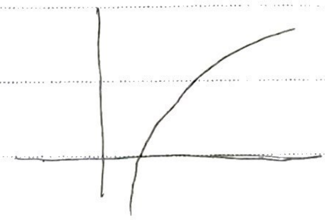
$7 \longrightarrow 11$

its 1-1 and onto and Isomorphic.

8. Show that the mapping $a \to \log_{10} a$ is an isomorphism for $\mathbb{R}^+$ under multiplication to $\mathbb{R}$ under addition.

$\phi : (\mathbb{R}^+, \cdot) \to (\mathbb{R}, +)$

$a \to \log_{10} a$



1-1 and onto.

① 1-1 :

let $\phi(a) = \phi(b)$ to prove $a = b$.

$\log_{10} a = \log_{10} b$

$\frac{\ln a}{\ln 10} = \frac{\ln b}{\ln 10} \rightarrow \ln a = \ln b$

$a = b$ so its 1-1.

② onto :

let $b \in \mathbb{R}$

→ If $b > 0$ then $\exists\, a > 1$ s.t $b = \log_{10} a$

$b = \phi(a)$.

→ If $b < 0$ then $\exists\; 0 < a < 1$ s.t $b = \log_{10} a = \phi(a)$.

→ If $b = 0$ then $0 = \log_{10} 1 = \log_{10} a = \phi(a)$, $a = 1$.

So $\phi$ is onto.

③ $\phi(xy) \stackrel{?}{=} \phi(x) + \phi(y)$.

$\phi(xy) = \log_{10}(xy)$

$= \log_{10} x + \log_{10} y$

$= \phi(x) + \phi(y)$ so its Isomorphism.

6. let $G$ be a group. Prove that the mapping $\alpha(g) = g^{-1}$ for all $g$ in $G$ is an automorphism iff $G$ is Abelian.

$\Rightarrow$ suppose $g_1, g_2 \in G$ then $\phi(g_1 g_2) = (g_1 g_2)^{-1}$

$= g_2^{-1} g_1^{-1}$ .... ①

and $\phi(g_1 g_2) = \phi(g_1)\phi(g_2) = g_1^{-1} g_2^{-1}$ .... ②

① and ② we have $g_2^{-1} g_1^{2} = g_1^{-1} g_2^{-1}$ $\forall g_1, g_2 \in G$.

$g_2 g_1 = g_1 g_2$ $\rightsquigarrow$ Abelian ✓

$\Leftarrow$ suppose $G$ is Abelian then $\phi(g) = g^{-1}$

① 1-1 : since $\phi(g_1) = \phi(g_2)$

implies $g_1^{-1} = g_2^{-1}$

$\rightsquigarrow$ $g_1 = g_2$ ✓

② onto : $\forall g \in G$, $\phi(g^{-1}) = (g^{-1})^{-1} = g$ ✓.

③ $\phi(g_1 g_2) = (g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$.

so $\phi(g_1 g_2) = (g_1 g_2)^{-1}$

$= g_2^{-1} g_1^{-1}$

$= g_1^{-1} g_2^{-1}$ since its Abelian

$= \phi(g_1)\phi(g_2)$

11. For Inner automorphisms $\phi_g, \phi_h, \phi_{gh}$, prove that $\phi_g \phi_h = \phi_{gh}$.

For any $x$ in the group:

$$\phi_g \phi_h(x) = \phi_g(\phi_h(x))$$
$$= \phi_g(hxh^{-1})$$
$$= ghxh^{-1}g^{-1}$$
$$= (gh)x(gh)^{-1}$$
$$= \phi_{gh}(x)$$

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12. Find two groups $G$ and $H$ such that $G \not\approx H$. But $Aut(G) \approx Aut(H)$.

let $G = \mathbb{Z}_6$ and $H = \mathbb{Z}_3$

$|G| = 6$ $|H| = 3$

$\rightarrow$ $G \not\cong H$ But $Aut(\mathbb{Z}_6) \cong U(6) = \{1, 5\}$

since $Aut(\mathbb{Z}_3) = U(3) = \{1, 2\}$

$\rightarrow$ Hence, $Auto(\mathbb{Z}_6) \cong Auto(\mathbb{Z}_3)$ since $U(6) \cong U(3)$.

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14. Find $Aut(Z_6)$:

$Aut(\mathbb{Z}_6) \cong U(6)$ $\phi(1) = 1$ or $\phi(1) = 5$.

$Aut(\mathbb{Z}_6) = \{\phi_1, \phi_5\}$.

15. If $G$ is a group, prove that $Aut(G)$ and $Inn(G)$ are groups.

let $\alpha \in Aut(G)$.

$\rightarrow$ $\alpha^{-1}(xy) = \alpha^{-1}(x)\alpha^{-1}(y)$ iff $\alpha(\alpha^{-1}(xy)) = \alpha(\alpha^{-1}(x)\alpha^{-1}(y))$

that is iff, $xy = \alpha(\alpha^{-1}(x))\alpha(\alpha^{-1}(y)) = xy$

So $\alpha^{-1}$ is operation-preserving.

That mean $Inn(G)$ is a group follows from the equation $\phi_g \phi_h = \phi_{gh}$.

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18. The group

$\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in \mathbb{Z} \right\}$ is Isomorphic to what familiar group? What if $\mathbb{Z}$ is replaced by $\mathbb{R}$?

its Isomorphic to $(\mathbb{Z}, +)$ and $\phi : G \to \mathbb{Z}$

$\phi\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\right) = a$ Is an isomorphism.

$H = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, a \in \mathbb{R} \right\} \cong (\mathbb{R}, +)$.

isomorphitic

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26. let $G = \{a + b\sqrt{2} : a, b$ Rational$\}$ and $H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} : a, b \text{ rational} \right\}$. ...

let $\phi : G \to H$

$a + b\sqrt{2} \to \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ $\phi$: Bijective.

Then $\phi((a + b\sqrt{2}) + (c + d\sqrt{2})) = \phi(a + c + (b+d)\sqrt{2})$

$$= \begin{bmatrix} a+c & 2(b+d) \\ b+d & a+c \end{bmatrix}$$

$$= \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} + \begin{bmatrix} c & 2d \\ d & c \end{bmatrix}$$

$$= \phi(a + b\sqrt{2}) + \phi(c + d\sqrt{2}).$$

And $\phi((a + b\sqrt{2}) \cdot (c + d\sqrt{2})) = \phi(a + b\sqrt{2}) \cdot \phi(c + d\sqrt{2})$.

So $\phi$ preserve multiplication.

27. Prove that $\mathbb{Z}$ under addition is not isomorphic to $Q$ under addition.

$$(\mathbb{Z}, +) \not\cong (Q, \cdot)$$

we need to show $Q$ is not cyclic.

suppose $Q$ is cyclic ( $\exists$ a rational number $r$ s.t $Q = \langle r \rangle$ ).

Then any rational number has the form $Kr$ for som $K$ ( $r^K = Kr$ ).

But $\frac{r}{2}$ is also rational number and there is No integer $K$ s.t $Kr = \frac{r}{2}$.

$\times$.

So $Q$ is not cyclic.

Since $(\mathbb{Z}, +)$ cyclic and $(Q, \cdot)$ Not cyclic so $(\mathbb{Z}, +) \not\cong (Q, \cdot)$.

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29. let $C$ be the complex numbers and

$M = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ prove that $C$ and $M$ are isomorphic under addition and

that $C^*$ and $M^*$, the nonzero elements of $M$, are isomorphic under multiplication.

$$\phi : C \longrightarrow M$$

$$a + bi \longrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$ it is Bijective.

$$\phi\left( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \right) = \phi\left( \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} \right)$$

$$(a+c) + (b+d)i = (a+bi) + (c+di) = \phi\left( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right) + \phi\left( \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \right).$$

So $\phi$ is isomorphism

$\rightarrow$ The $C^*$ and $M^*$ The same $\uparrow$.