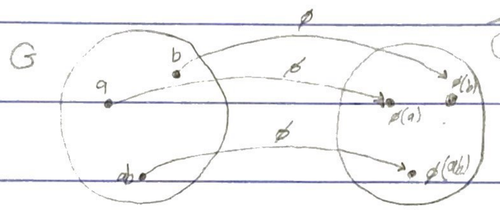
Chapter 6: Isomorphisms.

Def : group Isomorphism

An isomorphism ϕ from a group G to a group $\bar{G}$ is one to one mapping (or function) from G onto $\bar{G}$ that preserves the group operation. That is

$$\phi(ab) = \phi(a)\,\phi(b) \quad \text{for all } a, b \in G.$$

If there is an isomorphism from G onto $\bar{G}$, we say that $\bar{G}$ and G are isomorphic and write $G \approx \bar{G}$.

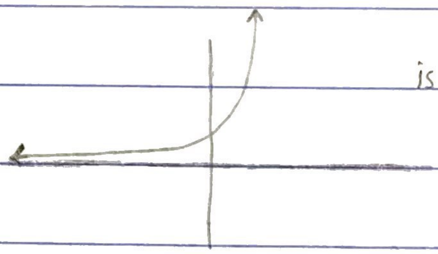


$\phi(ab) = \phi(a)\,\phi(b)$

exp: $G' = (\mathbb{R}, +)$, $H = (\mathbb{R}^+, \cdot)$

$\phi : G \to H$

$x \to e^x$



is bijective

$$\forall g_1, g_2 \in G , \quad \phi(g_1 + g_2) = \phi(g_1) \cdot \phi(g_2)$$

$$e^{g_1 + g_2} \stackrel{?}{=} e^{g_1} \cdot e^{g_2} \quad ✓$$

So its isomorphic.

exp : $G = (\mathbb{Z}_5^*, \otimes_5)$, $H = (U(10), \otimes_{10})$

$1 \longrightarrow 1$

$\phi: 2 \longrightarrow 3$ bijection

$3 \longrightarrow 7$

$4 \longrightarrow 9$ is isomorphic ?

$\rightarrow \phi(2.3) \overset{?}{=} \phi(2)\,\phi(3)$

$\phi(6) \overset{?}{=} (3)(7) \rightarrow$ Mod

$1 \overset{\checkmark}{=} 1$

$\rightarrow \phi(3.4)^{\mathbb{Z}_5} \overset{?}{=} \phi(3)\,\phi(4)$

$\phi(12)$ on $\mathbb{Z}_5 \rightarrow \phi(2) \overset{?}{=} (7)(9) \rightarrow U(10)$

$3 = 3$ ✓

$\rightarrow \phi(2.4) \overset{?}{=} \phi(2)\,\phi(4)$

$\phi(3) \overset{?}{=} (3)(9)$

$7 \overset{?}{=} 7$ ✓

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So its isomorphic.

exp1 on book.

$$(\mathbb{R}, +) \longrightarrow (\mathbb{R}^+, \cdot)$$

$$x \longrightarrow 2^x$$

$$\phi(x+y) \stackrel{?}{=} 2^{x+y}$$

$$= 2^x \cdot 2^y$$

$$= \phi(x)\,\phi(y) \quad \text{so isomorphic}$$

exp 2 :

$G = \langle a \rangle$, $|G| = \infty$

$(\mathbb{Z}, +) \approx G$

let $\phi : \mathbb{Z}, + \longrightarrow G$

$$K \longrightarrow a^K$$

$\phi(m+n) = a^{m+n} = a^m \cdot a^n$

$= \phi(m) \cdot \phi(n)$

so its isomorphic

Note : Any infinite cyclic group is isomorphic for ($\mathbb{Z}$ under $+$).

exp 2:

$$\phi : (\mathbb{R}, +) \longrightarrow (\mathbb{R}, +)$$

$$x \longrightarrow x^3$$

$$\phi(x+y) = (x+y)^3$$

$$\neq x^3 + y^3 = \phi(x) + \phi(y)$$

So $\phi(x+y) \neq \phi(x) + \phi(y)$ so its not isomorphitic.

exp: $(\mathbb{Z}_4, +)$, $(U(10), \odot_{10})$

let $\phi =$ $0 \longrightarrow 1$

$1 \longrightarrow 3$ is isomorphism

$2 \longrightarrow 9$

$3 \longrightarrow 7$

$\psi =$ $0 \longrightarrow 1$

$1 \longrightarrow 3$ Not isomorphism Because

$2 \longrightarrow 7$ $\psi(2+2) \neq \psi(2)\psi(2)$

$3 \longrightarrow 9$ $\psi(0) \neq (7)(7)$

$1 \neq 9$

Thm 6.1: Cayley's Theorem

every group is isomorphic to a group of permutations.

Thm 6.2: Properties of isomorphisms Acting on elements.

suppose that ϕ is an isomorphism from a group G onto a group $\overline{G}$, Then

1. ϕ carries the identity of G to the identity of $\overline{G}$.

2. For every integer n and for every group element a in G, $\phi(a^n) = [\phi(a)]^n$.

3. For any elements a and b in G, a and b commute iff $\phi(a)$ and $\phi(b)$ commute.

4. $G = \langle a \rangle$ iff $\overline{G} = \langle \phi(a) \rangle$.

5. $|a| = |\phi(a)|$ $\forall a \in G$ (isomorphisms preserve orders).

6. For a fixed integer K and a fixed group element b in G, the equation $x^K = b$ has the same number of solutions in G as does the equation $x^K = \phi(b)$ in $\overline{G}$.

7. If G is finite, then G and $\overline{G}$ have exactly the same number of elements of every order.

Def: Automorphism:

An isomorphism from a group G onto itself is called an automorphism of G.

→ $\phi : G \to G$ is isomorphism

means ϕ is automorphism of G.

→ $\phi : \mathbb{Z} \to \mathbb{Z}$ is isomorphism

$m \to -m$

$\phi(a+b) = -(a+b) = -a + -b$

$= \phi(a) + \phi(b)$

Def: Inner Automorphism Induced by a.

let G be a group and let $a \in G$. The function ϕ_a defined by $\phi_a(x) = axa^{-1}$ for all x in G is called the Inner automorphism of G induced by a.

$\phi : G \to G$

$x \to axa^{-1}$ is isomorphism ✓

$\phi(xy) = axya^{-1}$

$= (ax^{-1}a^{-1})(aya^{-1})$

$= \phi(x)\,\phi(y)$

So Its Inner Automorphism.

Thm 6.4: Aut(G) and Inn(G) are Groups.

The set of automorphisms of a group and the set of inner automorphisms of a group are both groups under the operation of function composition.

proof (⇒) $Aut(G) = \{ \phi : G \to G : \phi \text{ is automorphism} \}$, $*$ → composition of mapping

① closure : $\phi : G \to G$, $\psi : G \to G$ are Automorphisim

then $\phi \circ \psi : G \to G$ is Automorphisim.

ϕ, ψ is bijective and automorphism

$$(\phi \circ \psi)(g_1 g_2) = \phi(\psi(g_1 g_2))$$
$$= \phi(\psi(g_1)\psi(g_2))$$
$$= \phi(\psi(g_1))\,\phi(\psi(g_2))$$
$$= \phi * \psi(g_1 g_2)$$

② Identity : $I : G \to G$
$g \to g$

③ Inverse : ϕ^{-1} is the inverse function

(⇐) $Inn(G) = \{ \phi(a) : a \in G \}$ $\phi_a : G \to G, x \to axa^{-1}$

$\phi_a \circ \phi_b : G \to G$

$$x \to \phi_a \circ \phi_b(x) = \phi_a(\phi_b(x)) = \phi_a(bxb^{-1})$$
$$= a(bxb^{-1})a^{-1}$$
$$= abx(ab)^{-1}$$
$$= \phi_{ab}.$$

$(\phi_a)^{-1}$ is $\phi_{a^{-1}}$, identity $\phi_e : x \to exe^{-1}$.

chapter 6 done