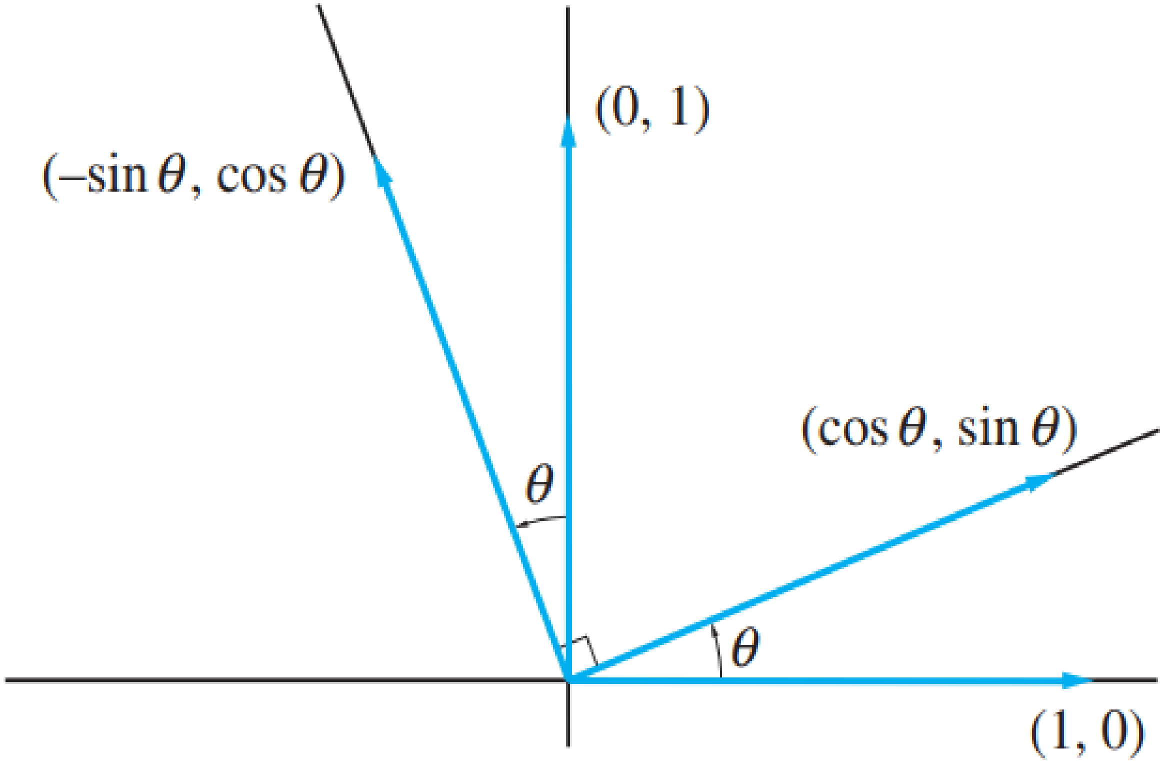
CHAPTER

4



Linear Transformations

4.1 Definition and Examples

In the study of vector spaces, the most important types of mappings are linear transformations.

Definition

A mapping L from a vector space V into a vector space W is said to be a **linear transformation** if

$$L(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = \alpha L(\mathbf{v}_1) + \beta L(\mathbf{v}_2) \tag{1}$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and for all scalars α and β.

Remark: L is a linear transformation if and only if L satisfies (2) and (3).

$$L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2) \qquad (\alpha = \beta = 1) \tag{2}$$

and

$$L(\alpha \mathbf{v}) = \alpha L(\mathbf{v}) \qquad (\mathbf{v} = \mathbf{v}_1, \beta = 0) \tag{3}$$

In the case that the vector spaces V and W are the same, we will refer to a linear transformation $L: V \to V$ as a *linear operator* on V. Thus, a linear operator is a linear transformation that maps a vector space V into itself.

Linear Operators on $\mathbb{R}^2$

EXAMPLE 1 Let L be the operator defined by

$$L(\mathbf{x}) = 3\mathbf{x}$$

for each $\mathbf{x} \in \mathbb{R}^2$. Since

$$L(\alpha\mathbf{x}) = 3(\alpha\mathbf{x}) = \alpha(3\mathbf{x}) = \alpha L(\mathbf{x})$$

and

$$L(\mathbf{x}+\mathbf{y}) = 3(\mathbf{x}+\mathbf{y}) = 3\mathbf{x}+3\mathbf{y} = L(\mathbf{x})+L(\mathbf{y})$$

it follows that L is a linear operator.

EXAMPLE 3 Let L be the operator defined by

$$L(\mathbf{x}) = (x_1, -x_2)^T$$

for each $\mathbf{x} = (x_1, x_2)^T$ in $\mathbb{R}^2$. Since

$$\begin{aligned} L(\alpha\mathbf{x} + \beta\mathbf{y}) &= \begin{pmatrix} \alpha x_1 + \beta y_1 \\ -(\alpha x_2 + \beta y_2) \end{pmatrix} \\ &= \alpha \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ -y_2 \end{pmatrix} \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \end{aligned}$$

it follows that L is a linear operator.

EXAMPLE 5 The mapping $L\colon \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$L(\mathbf{x}) = x_1 + x_2$$

is a linear transformation, since

$$\begin{aligned} L(\alpha\mathbf{x} + \beta\mathbf{y}) &= (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) \\ &= \alpha(x_1 + x_2) + \beta(y_1 + y_2) \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \end{aligned}$$

■

EXAMPLE 7 The mapping L from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = (x_2, x_1, x_1 + x_2)^T$$

is linear, since

$$L(\alpha\mathbf{x}) = (\alpha x_2, \alpha x_1, \alpha x_1 + \alpha x_2)^T = \alpha L(\mathbf{x})$$

and

$$\begin{aligned} L(\mathbf{x}+\mathbf{y}) &= (x_2 + y_2, x_1 + y_1, x_1 + y_1 + x_2 + y_2)^T \\ &= (x_2, x_1, x_1 + x_2)^T + (y_2, y_1, y_1 + y_2)^T \\ &= L(\mathbf{x}) + L(\mathbf{y}) \end{aligned}$$

In general, if A is any $m \times n$ matrix, we can define a linear transformation L_A from $\mathbb{R}^n$ to $\mathbb{R}^m$ by

$$L_A(\mathbf{x}) = A\mathbf{x}$$

for each $\mathbf{x} \in \mathbb{R}^n$. The transformation L_A is linear, since

$$\begin{aligned} L_A(\alpha\mathbf{x} + \beta\mathbf{y}) &= A(\alpha\mathbf{x} + \beta\mathbf{y}) \\ &= \alpha A\mathbf{x} + \beta A\mathbf{y} \\ &= \alpha L_A(\mathbf{x}) + \beta L_A(\mathbf{y}) \end{aligned}$$

Thus, we can think of each $m \times n$ matrix A as defining a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Linear Transformations from V to W

EXAMPLE 9 Let L be the mapping from $C[a, b]$ to $\mathbb{R}^1$ defined by

$$L(f) = \int_a^b f(x)\,dx$$

If f and g are any vectors in $C[a, b]$, then

$$\begin{aligned} L(\alpha f + \beta g) &= \int_a^b (\alpha f + \beta g)(x)\,dx \\ &= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx \\ &= \alpha L(f) + \beta L(g) \end{aligned}$$

Therefore, L is a linear transformation. ■

EXAMPLE 10 Let D be the linear transformation mapping $C^1[a, b]$ into $C[a, b]$ defined by

$$D(f) = f' \qquad \text{(the derivative of } f\text{)}$$

D is a linear transformation, since

$$D(\alpha f + \beta g) = \alpha f' + \beta g' = \alpha D(f) + \beta D(g)$$

Remark:

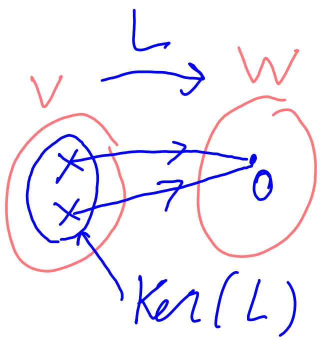
If L is a linear transformation mapping a vector space V into a vector space W, then

(i) $L(\mathbf{0}_V) = \mathbf{0}_W$ (where $\mathbf{0}_V$ and $\mathbf{0}_W$ are the zero vectors in V and W, respectively).

(ii) if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are elements of V and $\alpha_1, \ldots, \alpha_n$ are scalars, then

$$L(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) = \alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_n L(\mathbf{v}_n)$$

The Image and Kernel



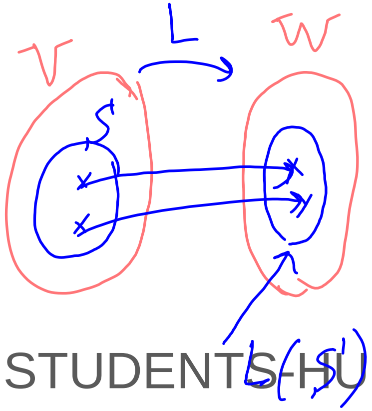
Let $L: V \to W$ be a linear transformation. We close this section by considering the effect that L has on subspaces of V. Of particular importance is the set of vectors in V that get mapped into the zero vector of W.

Definition

Let $L: V \to W$ be a linear transformation. The **kernel** of L, denoted $\ker(L)$, is defined by

$$\ker(L) = \{\mathbf{v} \in V \mid L(\mathbf{v}) = \mathbf{0}_W\}$$

Definition



Let $L: V \to W$ be a linear transformation and let S be a subspace of V. The **image** of S, denoted $L(S)$, is defined by

$$L(S) = \{\mathbf{w} \in W \mid \mathbf{w} = L(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in S\}$$

The image of the entire vector space, $L(V)$, is called the **range** of L.

Theorem 4.1.1 *If $L: V \to W$ is a linear transformation and S is a subspace of V, then*

(i) $\ker(L)$ *is a subspace of* V.

(ii) $L(S)$ *is a subspace of* W.

Theorem:

Let $T : V \to W$ be a linear transformation where V, W are vector spaces.

Suppose the dimension of V is n. Then $n = \dim(\ker(T)) + \dim(\mathrm{im}\,(T))$.

EXAMPLE 11 Let L be the linear operator on $\mathbb{R}^2$ defined by

$$L(\mathbf{x}) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

A vector $\mathbf{x}$ is in ker(L) if and only if $x_1 = 0$. Thus, ker(L) is the one-dimensional subspace of $\mathbb{R}^2$ spanned by $\mathbf{e}_2$. A vector $\mathbf{y}$ is in the range of L if and only if $\mathbf{y}$ is a multiple of $\mathbf{e}_1$. Hence, $L(\mathbb{R}^2)$ is the one-dimensional subspace of $\mathbb{R}^2$ spanned by $\mathbf{e}_1$. ■

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

(i) Ker L :

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \text{Ker } L \iff L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x_1 = 0$$

$$\therefore \text{Ker } L = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} : x_2 \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \text{span}\{e_2\}$$

(ii) Rang $L = L(\mathbb{R}^2)$

Let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, then $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

$$\therefore \text{Rang } L = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} : x_1 \in \mathbb{R}^2 \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \text{span}\{e_1\}$$

EXAMPLE 12 Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$L(\mathbf{x}) = (x_1 + x_2, x_2 + x_3)^T$$

and let S be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{e}_1$ and $\mathbf{e}_3$.

If $\mathbf{x} \in \ker(L)$, then

$$x_1 + x_2 = 0 \qquad \text{and} \qquad x_2 + x_3 = 0$$

Setting the free variable $x_3 = a$, we get

$$x_2 = -a, \qquad x_1 = a$$

and hence $\ker(L)$ is the one-dimensional subspace of $\mathbb{R}^3$ consisting of all vectors of the form $a(1, -1, 1)^T$.

If $\mathbf{x} \in S$, then $\mathbf{x}$ must be of the form $(a, 0, b)^T$, and hence $L(\mathbf{x}) = (a, b)^T$. Clearly, $L(S) = \mathbb{R}^2$. Since the image of the subspace S is all of $\mathbb{R}^2$, it follows that the entire range of L must be $\mathbb{R}^2$ [i.e., $L(\mathbb{R}^3) = \mathbb{R}^2$]. ■

$$L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

(i) Ker L:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \text{Ker } L \iff L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff x_3 = \alpha \;,\; x_2 = -\alpha \;,\; x_1 = \alpha$$

$$\therefore \text{Ker}(L) = \left\{ \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

(ii) $L(S')$ where $S' = \text{span}\{e_1, e_3\} = \{a e_1 + b e_3 : a, b \in \mathbb{R}\}$

$$= \left\{ \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Let $\begin{pmatrix} a \\ 0 \\ b \end{pmatrix} \in S'$, then $L(S') = \begin{pmatrix} a+0 \\ 0+b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\therefore L(S') = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\} = \mathbb{R}^2$

(iii) Rang $L = L(\mathbb{R}^3)$:

$$S' \subseteq \mathbb{R}^3 \Rightarrow L(S) \subseteq L(\mathbb{R}^3)$$

$$\Rightarrow \mathbb{R}^2 = L(S) \subseteq L(\mathbb{R}^3) \subseteq \mathbb{R}^2$$

$$\Rightarrow L(S') = L(\mathbb{R}^3) = \mathbb{R}^2.$$

SECTION 4.1 EXERCISES

4. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If

$$L((1,2)^T) = (-2,3)^T$$

and

$$L((1,-1)^T) = (5,2)^T$$

find the value of $L((7,5)^T)$.

6. Determine whether the following are linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^3$.

(a) $L(\mathbf{x}) = (x_1, x_2, 1)^T$

(b) $L(\mathbf{x}) = (x_1, x_2, x_1 + 2x_2)^T$

9. Determine whether the following are linear transformations from P_2 to P_3.

(a) $L(p(x)) = xp(x)$

(b) $L(p(x)) = x^2 + p(x)$

(c) $L(p(x)) = p(x) + xp(x) + x^2p'(x)$

(4) $\begin{pmatrix} 7 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for some c_1 and c_2.

Since $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

$L\begin{pmatrix} 7 \\ 5 \end{pmatrix} = c_1 L\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 L\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ since L is a linear transform.

$= c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

One can show that $c_1 = 4$ and $c_2 = 3$ (How?)

Then $L\begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} -8 + 15 \\ 12 + 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 18 \end{pmatrix}$

(6) (a) L is not a linear transformation since

$L\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

7 b L is not a linear transformation

$$L(0) = x^2 \neq 0$$

c i $L(cP(x)) = cP(x) + x(cP(x)) + x^2(cP(x))'$

$$= c\{P(x) + xP(x) + x^2 P'(x)\}$$

$$= c\,L(P(x))$$

ii $L(P(x)+q(x)) = (P(x)+q(x)) + x(P(x)+q(x)) + x^2(P(x)+q(x))'$

$$= (P(x) + xP(x) + x^2 P'(x)) + (q(x) + xq(x) + x^2 q'(x))$$

$$= L(P(x)) + L(q(x)).$$

Hence, L is a linear transformation

14. Let L be a linear operator on $\mathbb{R}^1$ and let $a = L(1)$. Show that $L(x) = ax$ for all $x \in \mathbb{R}^1$.

17. Determine the kernel and range of each of the following linear operators on $\mathbb{R}^3$:

(a) $L(\mathbf{x}) = (x_3, x_2, x_1)^T$ **(b)** $L(\mathbf{x}) = (x_1, x_2, 0)^T$

(c) $L(\mathbf{x}) = (x_1, x_1, x_1)^T$

21. A linear transformation $L: V \to W$ is said to be *one-to-one* if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$ (i.e., no two distinct vectors $\mathbf{v}_1, \mathbf{v}_2$ in V get mapped into the same vector $\mathbf{w} \in W$). Show that L is one-to-one if and only if $\ker(L) = \{\mathbf{0}_V\}$.

22. A linear transformation $L: V \to W$ is said to map V *onto* W if $L(V) = W$. Show that the linear transformation L defined by

$$L(\mathbf{x}) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)^T$$

maps $\mathbb{R}^3$ onto $\mathbb{R}^3$.

23. Which of the operators defined in Exercise 17 are one-to-one? Which map $\mathbb{R}^3$ onto $\mathbb{R}^3$?

(14) $L : \mathbb{R} \to \mathbb{R}$

$L(1) = a$

$L(x) = L(x(1)) = x\,L(1) = xa \quad \forall x \in \mathbb{R}$

(17) (a) $L : \mathbb{R}^3 \to \mathbb{R}^3$

(i) $\ker L$:

$(x_1, x_2, x_3)^T \in \ker L \iff L(x_1, x_2, x_3)^T = (0,0,0)^T \iff (x_3, x_2, x_1)^T = (0,0,0)^T$

$\iff x_1 = x_2 = x_3 = 0$

$\therefore \ker L = \{(0,0,0)^T\}$

$\therefore$ L is one to one

(ii) Range $L = L(\mathbb{R}^3)$

Let $(x_1, x_2, x_3)^T \in \mathbb{R}^3$, then

$L(x_1, x_2, x_3)^T = (x_3, x_2, x_1)^T = x_1 (0,0,1)^T + x_2 (0,1,0)^T + x_3 (1,0,0)^T$

$\therefore L(\mathbb{R}^3) = \text{span}\{e_1, e_2, e_3\} = \mathbb{R}^3$

$\therefore L$ is onto.

(c) $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

(a) Ker L :

$(x_1, x_2, x_3)^T \in \text{Ker} L \iff L(x_1, x_2, x_3)^T = (0,0,0)^T \iff (x_1, x_1, x_1)^T = (0,0,0)^T$

$\iff x_1 = 0$

$\therefore \operatorname{Ker} L = \{ (0, x_2, x_3)^T : x_2, x_3 \in \mathbb{R} \}$

$\therefore$ L is not one to one.

(b) Range L :

Let $(x_1, x_2, x_3)^T \in \mathbb{R}^3$, then

$$L(x_1, x_2, x_3)^T = (x_1, x_1, x_1)^T = x_1 (1, 1, 1)^T$$

$$\therefore L(\mathbb{R}^3) = \operatorname{span}\{ (1, 1, 1)^T \}$$

$\therefore$ L is not onto

