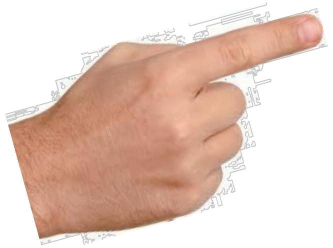
Sequences
& Mathematical Induction

Mustafa Jarrar



Watch this lecture
and download the slides

http://jarrar-courses.blogspot.com/2014/03/discrete-mathematics-course.html

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

Sequences & Mathematical Induction

5.1 Sequences

In this lecture:

- ❑ Part 1: **Why we need Sequences (Real-life examples).**
- ❑ Part 2: Sequence and Patterns
- ❑ Part 3: Summation: Notation, Expanding & Telescoping
- ❑ Part 4: Product and Factorial
- ❑ Part 5: Properties of Summations and Products
- ❑ Part 6: Sequence in Computer Loops and Dummy Variables

Keywords: Sequences, patterns, Summation, Telescoping, Product, Factorial, Dummy variables,

Motivation

هل يمكن النظر الى علم الرياضيات كعلم اكتشاف انماط في الحياة وتعميم
!!هذه الانماط كنظريات وقوانين؟
!!ما هو المشترك بين الفن وعلم الرياضيات؟

A mathematician, like a painteror poet, is a maker of patterns.

-G. H. Hardy, *A Mathematicians Apology, 1940*

Sequences (المتتاليات)

؟(حتى المستوى الاول :كم عدد أجدادك

؟(حتى المستوى الثاني)

؟(حتى المستوى الثالث)

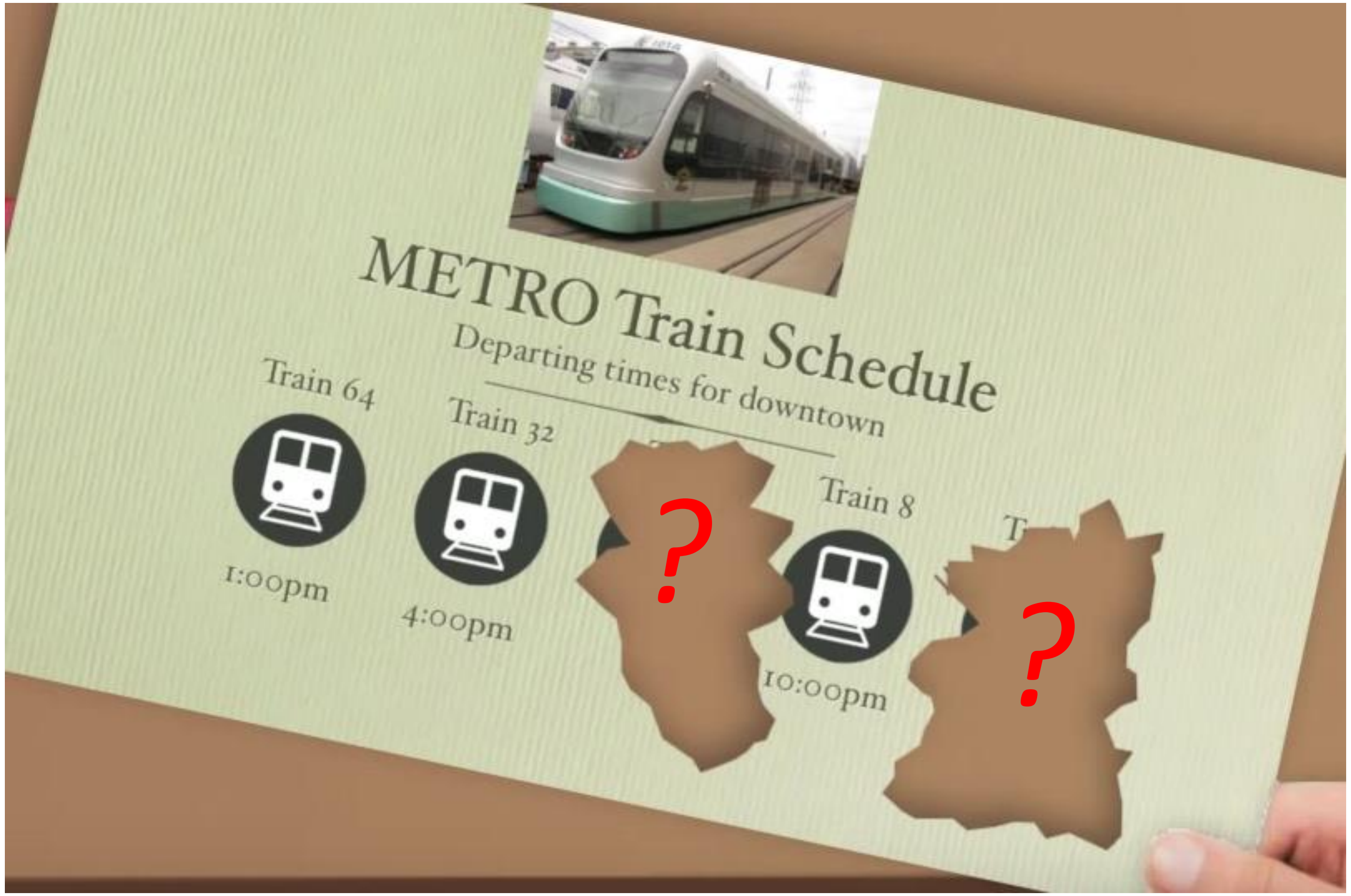
؟(حتى المستوى اخامس)

Position in the row	1	2	3	4	5	6	7 . . .
Number of ancestors	2	4	8	16	32	64	128 . . .

؟(K حتى المستوى)

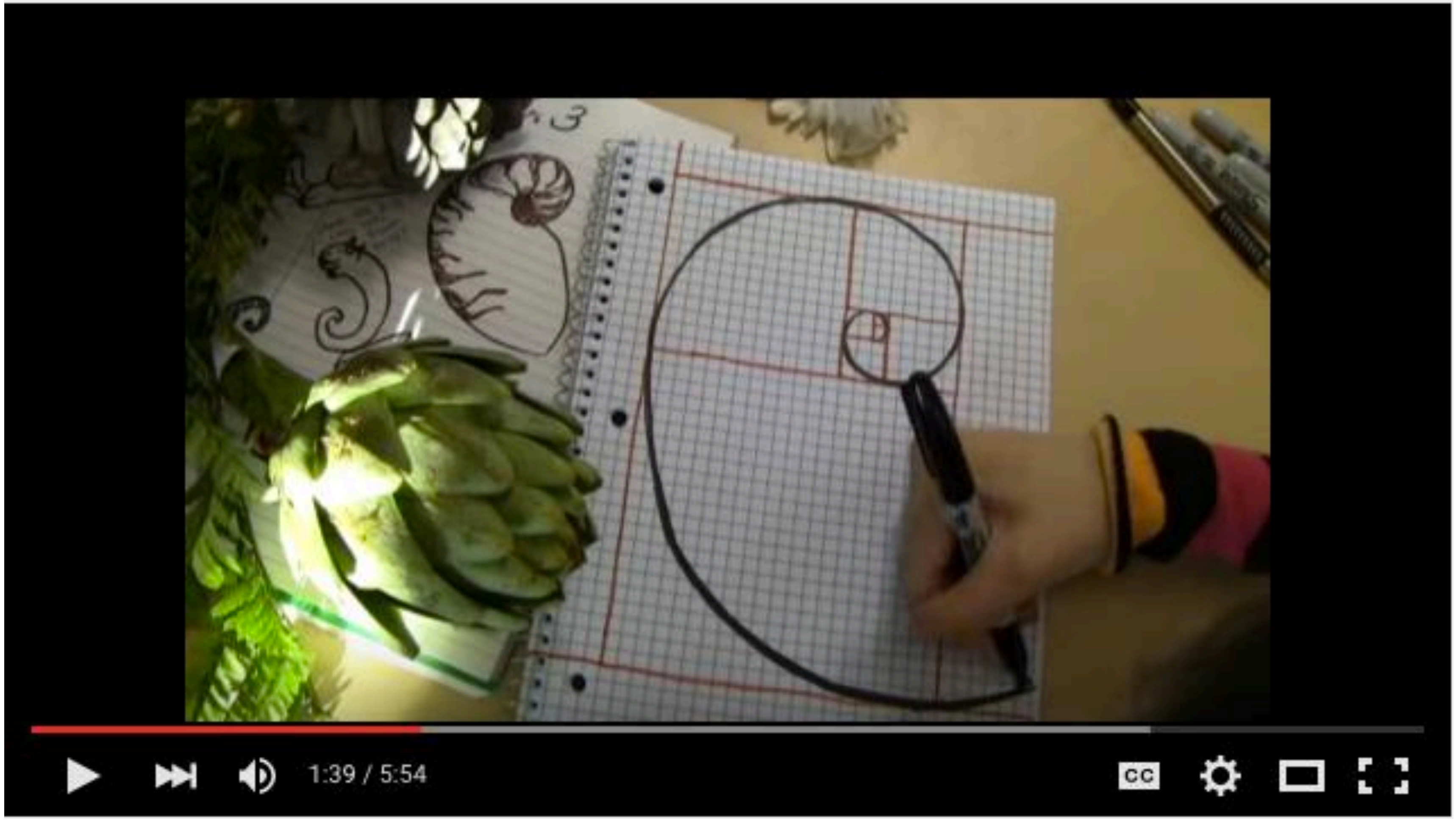
$$A_k = 2^K$$

Train Schedule



The Golden Ratio
$\phi = 1.618$

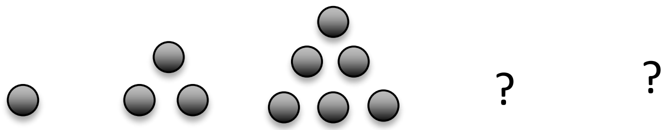
In Nature



https://www.youtube.com/watch?v=ahXIMUkSXX0

IQ Tests

Determine the number of points in the 4th and 5th figure



1 2 3 4 5

$a_n = \frac{n(n+1)}{2}$

Triangular Sequence

In programing

Any difference between these loops

1. **for** $i := 1$ **to** n
   print $a[i]$
   next i

2. **for** $j := 0$ **to** $n - 1$
   print $a[j + 1]$
   next j

3. **for** $k := 2$ **to** $n + 1$
   print $a[k - 1]$
   next k

$\sum_{k=1}^{n} a[k].$

$s := a[1]$
for $k := 2$ **to** n
  $s := s + a[k]$
next k

$s := 0$
for $k := 1$ **to** n
  $s := s + a[k]$
next k

Sequences & Mathematical Induction

5.1 Sequences

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Keywords: Sequences, patterns, Summation, Telescoping, Product, Factorial, Dummy variables,

Sequences
(المتتاليات)

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n$$

a Sequence is a set of elements written in a row.

Each individual element a_k is called a **term.**

The k in a_k is called a **subscript** or **index**

Finding Terms of Sequences Given by Explicit Formulas

Define sequences $a_1, a_2, a_3, \ldots$ and $b_2, b_3, b_4, \ldots$ by the following explicit formulas:

$$a_k = \frac{k}{k+1} \quad \text{for some integers } k \geq 1$$

$$b_i = \frac{i-1}{i} \quad \text{for some integers } i \geq 2$$

Compute the first five terms of both sequences.

Solution

ترتيب الحل وترتيب الافكار مهم جداً لملاحظة الانماط

$$a_1 = \frac{1}{1+1} = \frac{1}{2} \qquad b_2 = \frac{2-1}{2} = \frac{1}{2}$$

$$a_2 = \frac{2}{2+1} = \frac{2}{3} \qquad b_3 = \frac{3-1}{3} = \frac{2}{3}$$

$$a_3 = \frac{3}{3+1} = \frac{3}{4} \qquad b_4 = \frac{4-1}{4} = \frac{3}{4}$$

$$a_4 = \frac{4}{4+1} = \frac{4}{5} \qquad b_5 = \frac{5-1}{5} = \frac{4}{5}$$

$$a_5 = \frac{5}{5+1} = \frac{5}{6} \qquad b_6 = \frac{6-1}{6} = \frac{5}{6}$$

Finding Terms of Sequences Given by Explicit Formulas

Compute the first six terms of the sequence $c_0, c_1, c_2, \ldots$ defined as follows: $c_j = (-1)^j$ for all integers $j \geq 0$.

Solution:

$c_0 = (-1)^0 = 1$

$c^1 = (-1)^1 = -1$

$c^2 = (-1)^2 = 1$

$c^3 = (-1)^3 = -1$

$c^4 = (-1)^4 = 1$

$c^5 = (-1)^5 = -1$

Finding an Explicit Formula to Fit Given Initial Terms

Find an explicit formula for a sequence that has the following initial terms:

$$1, \quad -\frac{1}{4}, \quad \frac{1}{9}, \quad -\frac{1}{16}, \quad \frac{1}{25}, \quad -\frac{1}{36}, \ldots\ldots$$

$$a_k = \frac{(-1)^{k+1}}{k^2} \quad \text{for all integers } k \geq 1.$$

OR

$$a_k = \frac{(-1)^k}{(k+1)^2} \quad \text{for all integers } k \geq 0.$$

➔How to prove such formulas of sequences?

Sequences & Mathematical Induction

5.1 Sequences

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Keywords: Sequences, patterns, Summation, Telescoping, Product, Factorial, Dummy variables,

Summation

Upper limit

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \ldots a_n$$

Lower limit

index

Summation

• Definition

If m and n are integers and $m \leq n$, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from k equals m to n of a-sub-k**, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We say that $a_m + a_{m+1} + a_{m+2} + \ldots + a_n$ is the **expanded form** of the sum, and we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

We call k the **index** of the summation, m the **lower limit** of the summation, and n the **upper limit** of the summation.

Example

Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$.
Compute the following:

a. $\sum_{k=1}^{5} a_k$ b. $\sum_{k=2}^{2} a_k$ c. $\sum_{k=1}^{2} a_{2k}$

Solution:

a. $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = (-2) + (-1) + 0 + 1 + 2 = 0$

b. $\sum_{k=2}^{2} a_k = a_2 = -1$

c. $\sum_{k=1}^{2} a_{2k} = a_{2\cdot1} + a_{2\cdot2} = a_2 + a_4 = -1 + 1 = 0$

Example

When the Terms of a Summation are Given by a Formula

Compute the following summation:

$$\sum_{k=1}^{5} k^2.$$

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Useful Operations

- Summation to Expanded Form
- Expanded Form to Summation
- Separating Off a Final Term
- Telescoping

➔ **These concepts are very important to understand computer loops**

Summation to Expanded Form

Write the following summation in expanded form:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1}.$$

Solution

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

Expanded Form to Summation

Express the following using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

Solution

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=0}^{n} \frac{k+1}{n+k}.$$

Separating Off a Final Term and Adding On a Final Term n

Rewrite $\sum_{i=1}^{n+1} \frac{1}{i^2}$ **by separating off the final term.**

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

Write $\sum_{k=0}^{n} 2^k + 2^{n+1}$ **as a single summation.**

$$\sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

Telescoping

A telescoping series is a series whose partial sums eventually only have a fixed number of terms after cancellation [wiki].

Example: $\sum_{i=1}^{n} i-(i+1)$

$$=(1-2)+(2-3)+...+(n-(n+1))$$

$$=1-(n+1)$$

$$=-n$$

This is very useful in programing:

```
S=0
For (i=1;i<=n;i++)
    S= S+ i-(i+1);
```

→

```
S = -n;
```

Telescoping

A telescoping series is a series whose partial sums eventually only have a fixed number of terms after cancellation [1].

Example: $$\sum_{k=1}^{n} \frac{1}{k(k+1)}$$

$$= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

```
S=0;
For (k=1;k<=n;k++)
    S=S+ 1/k*(k+1);
```

```
S = 1- (1/(n+1);
```

Sequences & Mathematical Induction

5.1 Sequences

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Keywords: Sequences, patterns, Summation, Telescoping, Product, Factorial, Dummy variables,

Product Notation

• Definition

If m and n are integers and $m \leq n$, the symbol $\prod_{k=m}^{n} a_k$, read the **product from k equals m to n of a-sub-k**, is the product of all the terms $a_m,\ a_{m+1},\ a_{m+2},\ \ldots,\ a_n$.

We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

$$\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120 \qquad \prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$$

Factorial Notation

• Definition

For each positive integer n, the quantity ***n* factorial** denoted ***n*!,** is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted 0!, is defined to be 1:

$n! = \prod_{k=1}^{n} k$

$$0! = 1.$$

0! =1

1! =1

2! = 2·1 = 2

3! =3·2·1=6

4! = 4·3·2·1 = 24

5! = 5·4·3·2·1 = 120

6! = 6·5·4·3·2·1 = 720

7! = 7·6·5·4·3·2·1 = 5,040

8! = 8·7·6·5·4·3·2·1
= 40,320

9! = 9·8·7·6·5·4·3·2·1
= 362,880

Factorial Notation

A recursive definition for factorial

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n\,.\,(n-1)! & \text{if } n \geq 1 \end{cases}$$

recursive call

```
int fact(int x)
{
  if(x==0)
    return 1;
  else if (x>0)
    return fact(x-1)*x;
}
```

$0! = 1$

$1! = 1$

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$

$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$

$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880$

Computing with Factorials

$$\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$$

$$\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

$$\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} = n+1$$

$$\frac{n!}{(n-3)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)!}{(n-3)!} = n \cdot (n-1) \cdot (n-2)$$

$$= n^3 - 3n^2 + 2n$$

Sequences & Mathematical Induction

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Properties of Summations and Products

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$

2. $$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$$ generalized distributive law

3. $$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k).$$

➔ Remember to apply these in programing Loops

Example

Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expressions as a single summation or product:

$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$$

$$\prod_{k=m}^{n} a_k \cdot \prod_{k=m}^{n} b_k$$

$$\begin{aligned}\sum_{k=m}^{n} a_k + 2\cdot\sum_{k=m}^{n} b_k &= \sum_{k=m}^{n}(k+1) + 2\cdot\sum_{k=m}^{n}(k-1)\\ &= \sum_{k=m}^{n}(k+1) + \sum_{k=m}^{n} 2\cdot(k-1)\\ &= \sum_{k=m}^{n}((k+1) + 2\cdot(k-1))\\ &= \sum_{k=m}^{n}(3k-1)\end{aligned}$$

$$\begin{aligned}\left(\prod_{k=m}^{n} a_k\right)\cdot\left(\prod_{k=m}^{n} b_k\right) &= \left(\prod_{k=m}^{n}(k+1)\right)\cdot\left(\prod_{k=m}^{n}(k-1)\right)\\ &= \prod_{k=m}^{n}(k+1)\cdot(k-1)\\ &= \prod_{k=m}^{n}(k^2-1)\end{aligned}$$

Sequences & Mathematical Induction

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,

Change of Variable

Observe: $\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$ $\quad\quad$ $\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2.$

Hence: $\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2.$

Also Observe:

$$\sum_{j=2}^{4} (j-1)^2 = (2-1)^2 + (3-1)^2 + (4-1)^2$$

$$= 1^2 + 2^2 + 3^2$$

$$= \sum_{k=1}^{3} k^2.$$

Replaced Index by any other symbol (called a **dummy variable**).

Programing Loops

Any difference between these loops

1. **for** $i := 1$ **to** n
   print $a[i]$
   next i

2. **for** $j := 0$ **to** $n - 1$
   print $a[j + 1]$
   next j

3. **for** $k := 2$ **to** $n + 1$
   print $a[k - 1]$
   next k

$\sum_{k=1}^{n} a[k].$

$s := a[1]$
for $k := 2$ **to** n
  $s := s + a[k]$
next k

$s := 0$
for $k := 1$ **to** n
  $s := s + a[k]$
next k

Change Variables

Transform the following summation by making the specified change of variable.

$$\sum_{k=0}^{6} \frac{1}{k+1}$$ Change variable $j = k+1$

For (k=0; k≤6; k++)
Sum = Sum + 1/(k+1)

$$\sum_{j=1}^{7} \frac{1}{j} = \sum_{k=1}^{7} \frac{1}{k}.$$

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{k=1}^{7} \frac{1}{k}$$

For (k=1; k≤7; k++)
Sum = Sum + 1/(k)

Change Variables

Transform the following summation by making the specified change of variable.

$$\sum_{k=1}^{n+1} \frac{k}{n+k}$$

For (k=1; k<=n+1; k++)
Sum = Sum + k/(n+k)

Change of variable: j = k - 1

$$\sum_{j=0}^{n} \frac{j+1}{n+(j+1)} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}$$

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}$$

For (k=0; k<=n; k++)
Sum = Sum + (k+1)/(n+k+1)

Programing Loops

All questions in the exams will be loops

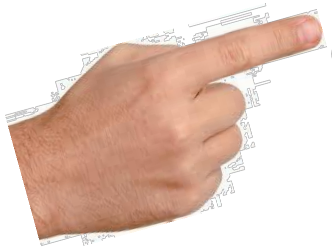
Thus, I suggest:

Convert all previous examples into loops and play with them

Sequences

& Mathematical Induction

5.1 Sequences



5.2&3 **Mathematical Induction (الاستقراء الرياضي)**

Sequences & Mathematical Induction

5.2&3 Mathematical Induction

In this lecture:

- ❑ **Part 1: What is Mathematical Induction**
- ❑ Part 2: Induction as a Method of Proof/Thinking
- ❑ Part 3: Proving *sum of integers* and *geometric sequences*
- ❑ Part 4: Proving a *Divisibility Property and Inequality*
- ❑ Part 5: Proving a *Property of a Sequence*
- ❑ Part 6: Induction Versus Deduction Thinking

What is Mathematical Induction

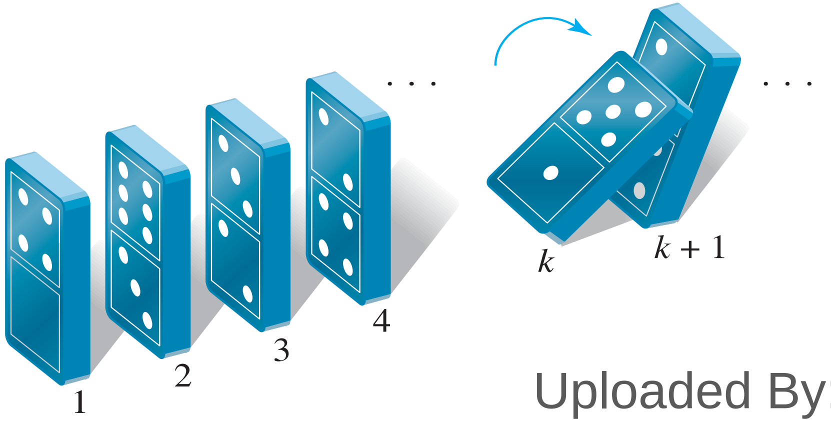
Mathematical induction is one of the more **recently developed methods of proof** in mathematics.

History:

The first use of mathematical induction was by الكرجي/Al-kraji (1000AD) in his book الفخري/ Al-Fakhri to prove math sequences. In 1883 Augustus De Morgan described it carefully and named mathematical induction.

The idea:

If the k^{th} domino falls backward, it pushes the $(k+1)^{st}$ domino backward.



What is Mathematical Induction

Principle of Mathematical Induction

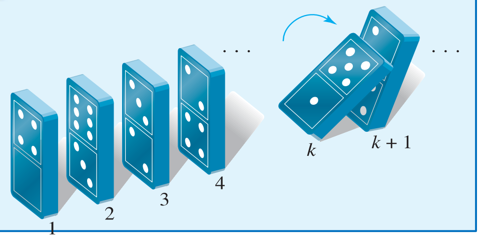
Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

for all integers $n \geq a$, $P(n)$

is true.



<u>Example:</u>

how to know whether this P(n) can be true?

P(n): For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.

Moves from specific cases to create a general rule (conjecture/ حدس), this is why it is called **Principle, not a theorem**

What is Mathematical Induction

Example

How to know whether this statement can be true?

For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.

For all integers $n \geq 8$, $P(n)$ is true, where $P(n)$ is the sentence "n cents can be obtained using 3¢ and 5¢ coins."

Then we need to prove that $P(n+1)$ is also true

Number of Cents	How to Obtain It
8¢	3¢ + 5¢
9¢	3¢ + 3¢ + 3¢
10¢	5¢ + 5¢
11¢	3¢ + 3¢ + 5¢
12¢	3¢ + 3¢ + 3¢ + 3¢
13¢	3¢ + 5¢ + 5¢
14¢	3¢ + 3¢ + 3¢ + 5¢
15¢	5¢ + 5¢ + 5¢
16¢	3¢ + 3¢ + 5¢ + 5¢
17¢	3¢ + 3¢ + 3¢ + 3¢ + 5¢

Sequences & Mathematical Induction

5.2&3 Mathematical Induction

In this lecture:

- Part 1: *What is Mathematical Induction*
- **Part 2: Induction as a Method of Proof/Thinking**
- Part 3: **Proving** *sum of integers* and *geometric sequences*
- Part 4: **Proving** a *Divisibility Property and Inequality*
- Part 5: **Proving** a *Property of a Sequence*
- Part 6: Induction Versus Deduction Thinking

Mathematical Induction as a Method of Proof

Proving a statement by mathematical induction is a two-step process. The first step is called the ***basis step***, and the second step is called the ***inductive step***.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis.**]

Then

show that $P(k+1)$ is true.

Mathematical Induction as a Method of Proof

Example

How to know whether this statement can be true?

For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.

Let the property $P(n)$ be the sentence: n¢ can be obtained using 3¢ and 5¢ coins. ← $P(n)$

Step 1 (basis step): ***Show P(8) is true:*** $P(8)$ is true as 8¢ obtained by one 3¢ and one 5¢

Step 2(inductive step): ***Show for all integers k ≥ 8, if P(k) is true then P(k+1) is true:***

[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 8. That is:]

Suppose k is any integer $k \geq 8$, k¢ obtained by 3¢ and 5¢ . ← $P(k)$ **inductive hypothesis**

[We must show that P(k + 1) is true. That is:] We must show that

$(k + 1)$ ¢ can be obtained using 3 ¢ and 5 ¢ coins. ← $P(k + 1)$

Case 1 (There is a 5¢ coin among those used to make up the k¢):

replace the 5¢ coin by two 3¢ coins; the result will be $(k + 1)$¢.

Case 2 (There is not a 5¢ coin among those used to make up the k ¢):

because $k \geq 8$, at least three 3¢ must have been used. So remove three 3¢ and replace them by two 5¢; the result will be $(k + 1)$¢.

Thus in either case $(k + 1)$¢ can be obtained using 3¢ and 5¢ *[as was to be shown]*.

Sequences & Mathematical Induction

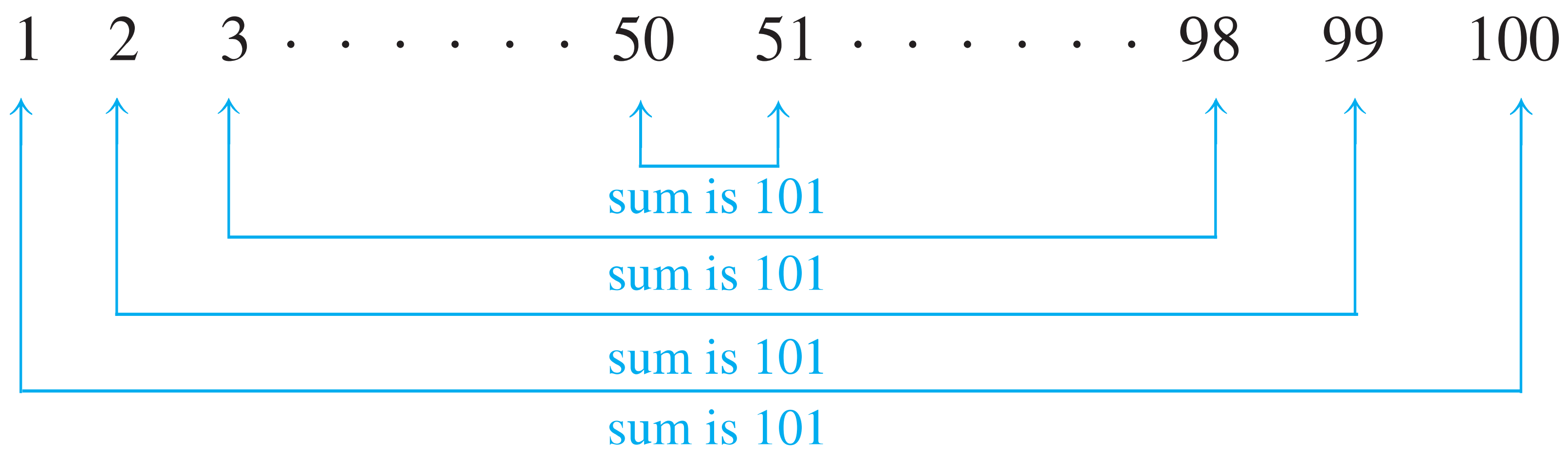
5.2&3 Mathematical Induction

In this lecture:

- Part 1: What is Mathematical Induction
- Part 2 : Induction as a Method of Proof/Thinking
- Part 3: **Proving Sum of Integers and Geometric Sequences**
- Part 4: Proving a *Divisibility Property and Inequality*
- Part 5: Proving a *Property of a Sequence*
- Part 6: Induction Versus Deduction Thinking

Sum of the First *n* Integers

Who can sum all numbers from 1 to 100?



$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Theorem 5.2.2 Sum of the First n Integers

For all integers $n \geq 1$, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Same Question: Prove that these programs prints the same results in case $n \geq 1$

```
For (i=1, i≤n; i++)
  S=S+i;
Print ("%d", S);
```

```
S=(n(n+1))/2
Print ("%d",S);
```

Proving that both programs produce the same results is like proving that:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \quad \leftarrow P(n)$$

Basis Step: *Show that P(1) is true.* P(1): 1 = 1(1+1)/2 = Thus P(1) is true

Inductive Step: *Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is also true:*

Suppose: $1+2+3+\ldots+k = \frac{k(k+1)}{2}$ is true $\leftarrow$ $P(k)$ inductive hypothesis

s

$P(k+1) = 1+2+\ldots+k + (k+1) = \frac{(k+1)(k+2)}{2}$ $\leftarrow P(k+1)$

$= P(k) + (k+1)$

$= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$= \frac{k^2+k}{2} + \frac{2(k+1)}{2} = \frac{k^2+3k+2}{2}$

$= \frac{(k+1)(k+2)}{2}$

Same

Examples of Sums

Evaluate $2 + 4 + 6 + \cdots + 500$.

$$\begin{aligned} 2+4+6+\cdots+500 &= 2\cdot(1+2+3+\cdots+250) \\ &= 2\cdot\left(\frac{250\cdot 251}{2}\right) \\ &= 62{,}750. \end{aligned}$$

Evaluate $5 + 6 + 7 + 8 + \cdots + 50$.

$$\begin{aligned} 5+6+7+8+\cdots+50 &= (1+2+3+\cdots+50)-(1+2+3+4) \\ &= \frac{50\cdot 51}{2} - 10 \\ &= 1{,}265 \end{aligned}$$

For an integer $h \geq 2$, write $1 + 2 + 3 + \cdots + (h-1)$ in closed form.

$$\begin{aligned} 1+2+3+\cdots+(h-1) &= \frac{(h-1)\cdot[(h-1)+1]}{2} \\ &= \frac{(h-1)\cdot h}{2} \end{aligned}$$

Theorem 5.2.3 Sum of a Geometric Sequence

For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Proof (by mathematical induction):

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1} - 1}{r - 1} \quad \leftarrow P(0) \quad = \frac{r-1}{r-1} = 1$$

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1} \quad \leftarrow P(k)$$

inductive hypothesis

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2} - 1}{r - 1}. \quad \leftarrow P(k+1)$$

$$= \sum_{i=0}^{k} r^i + r^{k+1}$$

$$= \frac{r^{k+1} - 1}{r - 1} + r^{k+1}$$

$$= \frac{r^{k+1} - 1}{r - 1} + \frac{r^{k+1}(r-1)}{r - 1}$$

$$= \frac{(r^{k+1} - 1) + r^{k+1}(r-1)}{r - 1}$$

$$= \frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r - 1}$$

$$= \frac{r^{k+2} - 1}{r - 1}$$

Mathematics in Programming

Example : Finding the sum of a geometric series

Prove that these codes will return the same output.

n.

```
int n, r, sum=0;
int i;
scanf("%d",&n);
scanf("%d",&r);

if(r != 1)  {
  for(i=0 ; i<=n ; i++)  {
       sum = sum + pow(r,i);
  }
  printf("%d\n“, sum);
}
```

```
int n, r, sum=0;
scanf("%d",&n);
scanf("%d",&r);

if(r != 1) {
       sum=((pow(r,n+1))-1)/(r-1);
       printf("%d\n", sum);
 }
```

This code is proposed by a student/Zaina!

Examples of Sums of a Geometric Sequence

In each of (a) and (b) below, assume that m is an integer that is greater than or equal to 3. Write each of the sums in closed form.

(a) $\mathbf{1+3+3^2+\cdots+3^{m-2}}$

$$1+3+3^2+\cdots+3^{m-2} = \frac{3^{(m-2)+1}-1}{3-1} = \frac{3^{m-1}-1}{2}.$$

(b) $\mathbf{3^2+3^3+3^4+\cdots+3^m}$

$$3^2+3^3+3^4+\cdots+3^m = 3^2\cdot(1+3+3^2+\cdots+3^{m-2}) = 9\cdot\left(\frac{3^{m-1}-1}{2}\right)$$

Sequences & Mathematical Induction

5.2&3 Mathematical Induction

In this lecture:

- Part 1: What is Mathematical Induction
- Part 2 : Induction as a Method of Proof/Thinking
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- Part 5: Proving a *Property of a Sequence*
- Part 6: Induction Versus Deduction Thinking

Proposition 5.3.1 Proving a Divisibility Property

For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Proof (by mathematical induction):

$3 \mid 2^{2n} - 1$ $\leftarrow P(n)$

Basis Step: ***Show that P(0) is true.***

P(0): $2^{2.0} - 1 = 2^0 - 1 = 1 - 1 = 0$ As $3 \mid 0$, thus $P(0)$ is true.

Inductive Step: ***Show that for all integers k ≥ 0, if P(k) is true then P(k + 1) is also true:***

Suppose: $2^{2k} - 1$ is divisible by 3. $\leftarrow P(k)$ inductive hypothesis

$2^{2k} - 1 = 3r$ for some integer r.

$2^{2(k+1)} - 1$ is divisible by 3. $\leftarrow P(k+1)$

$2^{2(k+1)} - 1 = 2^{2k+2} - 1$ by the laws of exponents

$= 2^{2k} \cdot 2^2 - 1 = 2^{2k} \cdot 4 - 1$

$= 2^{2k}(3 + 1) - 1 = 2^{2k} \cdot 3 + (2^{2k} - 1) = 2^{2k} \cdot 3 + 3r$

$= 3(2^{2k} + r)$ Which is integer

so, by definition of divisibility, $2^{2(k+1)} - 1$ is divisible by 3

 ,

Mathematics in Programming
Example : Proving Property of a Sequence

What will the output of this program be for any input n?

```
int n;
scanf("%d",&n);

if(n >= 0) {
    if( (pow(2,(2*n))  - 1) %3 == 0)
      printf("this property is true");
    else
      printf("this property isn't true");
}
```

$3 \mid 2^{2n} - 1$?

Mathematics in Programming

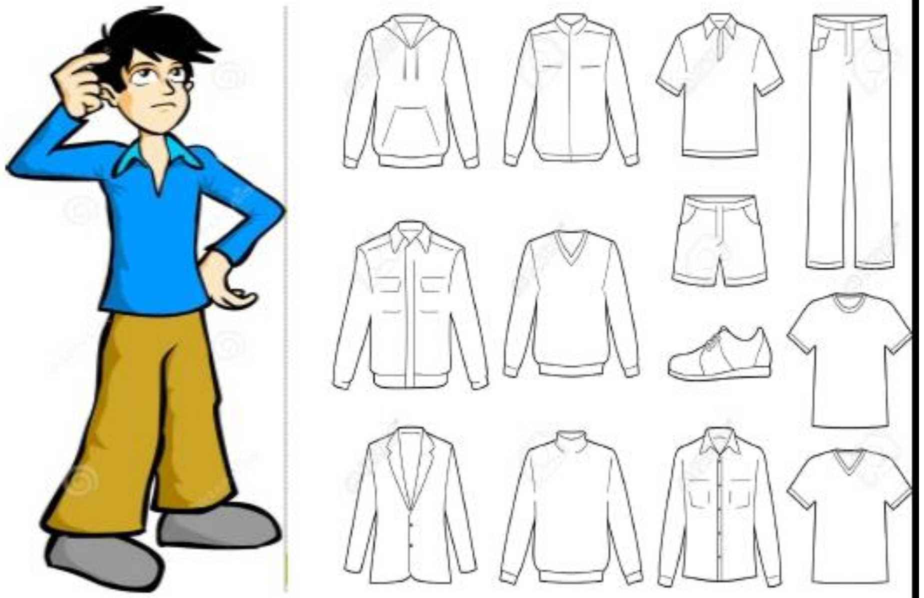
Example : Proving Property of a Sequence

What will this guy choose to wear today ?

(What is the output of the program)

```
int x, y;
scanf("%d %d", &x, &y);
if(x%2 == 0)
    x=x+1;
if(pow(x, 2)%2 != 0)
    printf("White Shirt");
else
    printf("Black Shirt");

if((pow(7, y)-1)%6==0)
    printf("Black boot");
else
    printf ("White Boot");
```



Proposition 5.3.2 Proving Inequality

For all integers $n \geq 3$, $2n + 1 < 2^n$.

Proof (by mathematical induction):

Let P(n) be $2n+1<2^n$

Basis Step: ***Show that P(3) is true.*** P(3): $2.3+1 < 2^3$ which is true.

Inductive Step: ***Show that for all integers k ≥ 3, if P(k) is true then P(k + 1) is also true:***

Suppose: $2k+1<2^k$ is true ← $P(k)$ inductive hypothesis

$2(k+1)+1 < 2^{k+1}$ ← $P(k+1)$

$2k+3 = (2k+1)+2$ by algebra

$(2k+1)+2 < 2^k + 2^k$ as $2k + 1 < 2^k$ by the inductive hypothesis and because $2 < 2^k$ $(k \geq 2)$

$\therefore 2k + 3 < 2 \cdot 2^k = 2^{k+1}$

[This is what we needed to show.]

Sequences & Mathematical Induction

5.2&3 Mathematical Induction

In this lecture:

- ❑ Part 1: What is Mathematical Induction
- ❑ Part 2 : Induction as a Method of Proof/Thinking
- ❑ Part 3: Proving Sum of Integers and Geometric Sequences
- ❑ Part 4: Proving a Divisibility Property and Inequality
- ❑ Part 5: **Proving a Property of a Sequence**
- ❑ Part 6: Induction Versus Deduction Thinking

Proving a Property of a Sequence

Example

Define a sequence a_1, a_2, a_3 . . . as follows:

$a_1 = 2$

$a_k = 5a_{k-1}$ for all integers $k \geq 2$.

Write the first four terms of the sequence.

$a_1 = 2$

$a_2 = 5a_{2-1} = 5a_1 = 5 \cdot 2 = 10$

$a_3 = 5a_{3-1} = 5a_2 = 5 \cdot 10 = 50$

$a_4 = 5a_{4-1} = 5a_3 = 5 \cdot 50 = 250$

➔ **The terms of the sequence satisfy the equation $a_n = 2 \cdot 5^{n-1}$**

Proving a Property of a Sequence

Example

Prove this property:

$$a_n = 2 \cdot 5^{n-1} \text{ for all integers } n \geq 1$$

Basis Step: *Show that P(1) is true.* $a_1 = 2 \cdot 5^{1-1} - 1 = 2 \cdot 5^0 - 1 = 2$

Inductive Step: *Show that for all integers k ≥ 1, if P(k) is true then P(k + 1) is also true:*

Suppose: $a_k = 2 \cdot 5^{k-1}$ ← $P(k)$ inductive hypothesis

$a_{k+1} = 2 \cdot 5^k$ ← $P(k+1)$

$= 5a_{(k+1)-1}$ by definition of $a_1, a_2, a_3 \ldots$

$= 5a_k$

$= 5 \cdot (2 \cdot 5^{k-1})$ by the hypothesis

$= 2 \cdot (5 \cdot 5^{k-1})$

$= 2 \cdot 5^k$

[This is what we needed to show.]

Sequences & Mathematical Induction

5.2&3 Mathematical Induction

In this lecture:

- ❑ Part 1: What is Mathematical Induction
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- ❑ Part 3: Proving Sum of Integers and Geometric Sequences
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- ❑ Part 5: Proving a Property of a Sequence
- ❑ Part 6: **Induction Versus Deduction Thinking**

Induction Versus Deduction Reasoning

Deduction Reasoning

If Every man is person and Sami is Man,
then Sami is Person

If my highest mark this semester is 82%, then my average will not be more than 82%

Induction Reasoning

For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.

We had a quiz each lecture in the past months, so we will have a quiz next lecture

Induction Versus Deduction Reasoning

Deduction Reasoning	Induction Reasoning
Based on facts, definitions, , theorems, laws	Based on observation, past experience, patterns
Moves from general observation to specific results	Moves from specific cases to create a general rule
Provides proofs	**Provides conjecture/حدس**

More slides from students

Student: Ehab, 2016

Not reviewed or verified

Example[1]

prove the following property:

for all integers $n \geq 1$, $1\times2 + 2\times3 + 3\times4 + \ldots + (n)(n+1) = \dfrac{(n)(n+1)(n+2)}{3}$

basis step : show p(1) is true. P(1): $1\text{x}2 = \dfrac{(1)(2)(3)}{3}$

left-hand side is $1\times2 = 2$

right-hand side is $\dfrac{(1)(2)(3)}{3} = 2$

thus p(1) is true

inductive step : Show that for all integers $k \geq 1$, if P(k) is true then P(k + 1) is also true:

suppose that p(k) is true

$p(k) = 1\times2 + 2\times3 + 3\times4 + \ldots + (k)(k+1) = \dfrac{(k)(k+1)(k+2)}{3}$ ← P(k) inductive hypothesis

$p(k+1)= 1\times2 + 2\times3 + 3\times4 + \ldots + (k)(k+1) + (k+1)((k+1)+1)$

$= [1\times2 + 2\times3 + 3\times4 + \ldots + (k)(k+1)] + (k+1)((k+1)+1)$

$= \dfrac{(k)(k+1)(k+2)}{3} + (k+1)(k+2)$

$= \dfrac{(k)(k+1)(k+2)}{3} + \dfrac{3(k+1)(k+2)}{3}$

$= \dfrac{(k+1)(k+2)(k+3)}{3}$ = right side *[This is what we needed to show.]*

Then p(k) works for all $n \geq 1$.

[1] CALCULUS with Analytic Geometry, Earl W.Swokwski

Example[1]

Show that For any integer $n \geq 5$, $4n < 2^n$.

basis step : show $P(n = 5)$ is true.

$4n = 4 \times 5 = 20$, and $2^n = 2^5 = 32$.

Since $20 < 32$, thus $p(n=5)$ is true

inductive step : Show that for all integers $k \geq 0$, if $p(k)$ is true then $p(k+1)$ is true:

suppose $p(k)$ is true for $k \geq 5$ ← $P(k)$ inductive hypothesis

$p(k+1)$: $4(k + 1) = 4k + 4$, and, by assumption $[4k] + 4 < [2^k] + 4$

Since $k \geq 5$, then $4 < 32 \leq 2^k$. Then we get

$2^k + 4 < 2^k + 2^k =$

$= 2 \times 2^k$

$= 2^1 \times 2^k$

$= 2^{k+1}$

Then $4(k+1) < 2^{k+1}$,hence $p(k+1)$ is true *[This is what we needed to show.]*

[1] question taken from this book: CALCULUS with Analytic Geometry, Earl W.Swokwski

Example[1]

show that For all $n \geq 1$, $8^n - 3^n$ is divisible by 5.

basis step : show that p(1) is true

$8^1 - 3^1 =$

$= 8 - 3$

$= 5$ which is clearly divisible by 5.

inductive step : Show that for all integers k>0 , if p(k) is true then p(pk+1) is true:

Suppose p(k) is true ($8^k - 3^k$ is divisible by 5) ← P(k) inductive hypothesis

$8^{k+1} - 3^{k+1} =$

$= 8^{k+1} - 3\times8^k + 3\times8^k - 3^{k+1}$

$= 8^k(8 - 3) + 3(8^k - 3^k)$

$= 8^k(5) + 3(8^k - 3^k)$

The first term in $8^k(5) + 3(8^k - 3^k)$ has 5 as a factor (explicitly), and the second term is divisible by 5 (by assumption). Since we can factor a 5 out of both terms, then the entire expression, $8^k(5) + 3(8^k - 3^k) = 8^{k+1} - 3^{k+1}$, must be divisible by 5.

[This is what we needed to show.]

[1] question taken from this book: CALCULUS with Analytic Geometry, Earl W.Swokwski

Example[1]

$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2 (n+1)^2}{4}$ .show that this equation is true for all integers n $\geq$1.

Basis step: show that p (1) is true.

Left Side = $1^3 = 1$

Right Side = $\frac{1^2 (1+1)^2}{4} = 1$

hence p (1) is true.

Inductive step: Show that for all integers k>0 , if p(k) is true then p(pk+1) is true:

suppose that p (k) is true ← P(k) inductive hypothesis

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3$

$= \frac{k^2 (k+1)^2}{4} + (k+1)^3$

$= \frac{k^2 (k+1)^2}{4} + \frac{4(k+1)^3}{4}$

$= \frac{(k+1)^2 [k^2 + 4k + 4]}{4}$

$= \frac{(k+1)^2 [(k+2)^2]}{4}$

= right side   *[This is what we needed to show.]*

[1] question taken from this book: CALCULUS with Analytic Geometry, Earl W.Swokwski

Example[1]

Prove that for any integer number $n \geq 1$, $n^3 + 2n$ is divisible by 3

Basis Step: show that p (1) is true.
Let n = 1 and calculate $n^3 + 2n$
$1^3 + 2(1) = 3$
3 is divisible by 3 ,hence p (1) is true.

Inductive Step: Show that for all integers k>0 , if p(k) is true then p(pk+1) is true:
suppose that p (k) is true ← P(k) inductive hypothesis
$(k + 1)^3 + 2(k + 1)$
$= k^3 + 3k^2 + 5k + 3$
$= [k^3 + 2k] + [3k^2 + 3k + 3]$
$= 3[k^3 + 2k] + 3[k^2 + k + 1]$
$= 3[[k^3 + 2k] + k^2 + k + 1]$
Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

[This is what we needed to show.]

[1] question taken from this book: CALCULUS with Analytic Geometry, Earl W.Swokowski