Key

Birzeit University
Mathematics Department
Math331-Section (1)
Second Short Exam

Instructor: Dr. Ala Talahmeh
Time: 50 minutes
Name:...

First Semester 2024/2025
Date: 27/11/2024
Number:..............

Question#1 [**4 marks**]. Consider the following DE

$$xy^2dx + (xy - y)dy = 0, \quad x > 1.$$

(a) Show that the above differential equation is **not** exact.

1

$M(x,y) = xy^2, \quad N(x,y) = xy - y.$

$M_y = 2xy, \quad N_x = y, \quad M_y \neq N_x.$

(b) Change the differential equation into exact. (**Do not solve it**).

1

$$\frac{M_y - N_x}{N} = \frac{2xy - y}{xy - y} = \frac{y(2x-1)}{y(x-1)} = \frac{2x-1}{x-1} = 2 + \frac{1}{x-1}.$$

1

$$I.F = e^{\int (2 + \frac{1}{x-1})dx} = e^{2x + \ln|x-1|} = (x-1)e^{2x}, \ x > 1.$$

1

The new exact eq. is $x(x-1)e^{2x}y^2dx + (x-1)^2ye^{2x}dy = 0$

Question#2 [**3 marks**]. Solve the initial-value problem:

$$y'' - 3y' - 4y = 0, \quad y(0) = 5, \quad y'(0) = 0.$$

Let $y = e^{rx}$. The aux. eq. is $r^2 - 3r - 4 = 0$

1

$(r-4)(r+1) = 0$

$r = 4, -1.$

1

$\therefore y = c_1e^{4t} + c_2e^{-t}$

$5 = y(0) = c_1 + c_2$ — (i)

$y' = 4c_1e^{4t} - c_2e^{-t}, \quad y'(0) = 4c_1 - c_2 = 0$ ... (ii)

0.5 0.5

(i) and (ii) give $c_1 = 1, \ C_2 = 4$

$\therefore y = e^{4t} + 4e^{-t}.$

Question#3 [6 marks].

(a) Find the **largest interval** for which the following IVP has a unique solution

$$(x-1)^3y'' + \ln(3-x)y' + \frac{1}{\sqrt{x}}y = e^x, \quad y(2)=0, \quad y'(2)=1.$$

1 $p(x) = \frac{\ln(3-x)}{(x-1)^3}$, $q(x) = \frac{1}{\sqrt{x}\,(x-1)^3}$, $g(x) = \frac{e^x}{(x-1)^3}$

$p, q,$ and g are continuous on $(0,1)\cup(1,3)$.

$\therefore$ the largest interval 1 is $(1,3)$.

(b) Solve the initial-value problem:

$$y'y'' = 2, \quad y(1) = \frac{4}{3}, \quad y'(1) = 2.$$

0.5 $v = y'$, $v' = y''$

So, $v v' = 2$ or 0.5 $\int v\,dv = \int 2\,dt$

0.5 $\frac{v^2}{2} = 2t + C_1$

$2 = \frac{(2)^2}{2} = \frac{v(1)^2}{2} = 2 + C_1 \Rightarrow$ 0.5 $C_1 = 0$

$\therefore v^2 = 4t$

0.5 $v = 2\sqrt{t}$ or $v = -2\sqrt{t}$ (reject)

0.5 $\int dy = \int 2\sqrt{t}\,dt$

0.5 $y = \frac{4}{3}t^{3/2} + C_2$

$\frac{4}{3} = y(1) = \frac{4}{3} + C_2 \Rightarrow$ 0.5 $C_2 = 0$

$\therefore \boxed{y = \frac{4}{3}t^{3/2}}$

Question#4 [**3 marks**]. Solve the following IVP using **Picard's method**

$$\frac{dy}{dt} = y + 1, \quad y(0) = 0.$$

$f(t,y) = y + 1.$

Take $\phi_0(t) = 0$, $\phi_1(t) = \int_0^t f(s,0)\,ds = \int_0^t 1\,ds = t$

$$\phi_2(t) = \int_0^t f(s,s)\,ds = \int_0^t (s+1)\,ds = \frac{t^2}{2} + t.$$

$$\phi_3(t) = \int_0^t f\left(s, \frac{s^2}{2} + s\right) ds$$

$$= \int_0^t \left(\frac{s^2}{2} + s + 1\right) ds = \frac{t^3}{3!} + \frac{t^2}{2!} + t.$$

$\vdots$

2 $\phi_n(t) = t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} = \sum_{k=1}^{n} \frac{t^k}{k!}.$

$\lim_{n\to\infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!}$ (conv. by the ratio test).

$\therefore\ y =$ 1 $\sum_{k=1}^{\infty} \frac{t^k}{k!} = e^t - 1$ is the solution of the given IVP.

Question#5 [**4 marks**]. Given that $y_1(x) = e^{-2x}$ is a solution of the DE

$$(2x+1)y'' + 4xy' - 4y = 0, \quad x > -\frac{1}{2}.$$

Use **reduction of order formula** to find a second solution $y_2(x)$.

$$y'' + \frac{4x}{2x+1}y' - \frac{4}{2x+1}y = 0, \quad x > -\frac{1}{2}.$$

$$p(x) = \frac{4x}{2x+1}.$$

1 $$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

$$= e^{-2x} \int \frac{e^{-\int \frac{4x}{2x+1}dx}}{e^{-4x}} dx$$

1 $$= e^{-2x} \int \frac{e^{-\int \left(2 - \frac{2}{2x+1}\right)dx}}{e^{-4x}} dx$$

$2x+1$ $\quad e^{2x}$

$(+)$

2 $\quad \frac{1}{2}e^{2x}$

$(-)$

0 $\quad \frac{1}{4}e^{2x}$

1 $$= e^{-2x} \int \frac{e^{-2x}(2x+1)}{e^{-4x}} dx$$

$$= e^{-2x} \int (2x+1)e^{2x} dx$$

1 $$= e^{-2x}\left(\frac{2x+1}{2}e^{2x} - \frac{1}{2}e^{2x} + c\right) = x + \boxed{ce^{-2x}} \text{ absorb.}$$

$$\therefore \boxed{y_2 = x}$$