Elementary Matrices

If we start with the identity matrix I and then perform exactly one elementary row operation, the resulting matrix is called an *elementary* matrix.

There are three types of elementary matrices corresponding to the three types of elementary row operations.

Type I An elementary matrix of type I is a matrix obtained by interchanging two rows of I.

EXAMPLE 1 The matrix

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an elementary matrix of type I since it was obtained by interchanging the first two rows of I. If A is a 3×3 matrix, then

$$E_1A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$AE_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{pmatrix}$$

Type II An elementary matrix of type II is a matrix obtained by multiplying a row of I by a nonzero constant.

EXAMPLE 2

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is an elementary matrix of type II. If A is a 3×3 matrix, then

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$$

$$AE_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 3a_{13} \\ a_{21} & a_{22} & 3a_{23} \\ a_{31} & a_{32} & 3a_{33} \end{bmatrix}$$

Type III An elementary matrix of type III is a matrix obtained from I by adding a multiple of one row to another row.

EXAMPLE 3

$$E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an elementary matrix of type III. If A is a 3×3 matrix, then

$$E_3A = \begin{bmatrix} a_{11} + 3a_{31} & a_{12} + 3a_{32} & a_{13} + 3a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$AE_3 = \begin{bmatrix} a_{11} & a_{12} & 3a_{11} + a_{13} \\ a_{21} & a_{22} & 3a_{21} + a_{23} \\ a_{31} & a_{32} & 3a_{31} + a_{33} \end{bmatrix}$$

EXERCISES

1. Which of the matrices that follow are elementary matrices? Classify each elementary matrix by type.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Theorem 1.5.1 *If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.*

EXERCISES

2. Find the inverse of each matrix in Exercise 1. For each elementary matrix, verify that its inverse is an elementary matrix of the same type.

Definition

A matrix B is **row equivalent** to a matrix A if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_1 A$$

The following properties of row equivalent matrices are easily established.

I. If A is row equivalent to B, then B is row equivalent to A.

II. If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.

Proof

EXERCISES

5. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{bmatrix}$$

(a) Find an elementary matrix E such that $EA = B$.

(b) Find an elementary matrix F such that $FB = C$.

(c) Is C row equivalent to A? Explain.

Theorem 1.5.2 **Equivalent Conditions for Nonsingularity**

Let A be an $n \times n$ matrix. The following are equivalent:

(a) A *is nonsingular.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution* $\mathbf{0}$.

(c) A *is row equivalent to* I.

Proof

Corollary 1.5.3 *The system $A\mathbf{x} = \mathbf{b}$ of n linear equations in n unknowns has a unique solution if and only if A is nonsingular.*

Proof

($\Leftarrow$) Let A be non singular.

$$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$$

($\Rightarrow$) Assume that $Ax = b$ has a unique solution $x = x_1$.

and assume that A is singular.

$\Rightarrow$ $Ax = 0$ has a non-zero solution $x = x_2$.

Let $y = x_1 + x_2$, then $Ay = A(x_1 + x_2) = Ax_1 + Ax_2$

$= b + 0 = b$

This is a contradiction

Hence A is non-singular.

EXERCISES

16. Let A be a 3×3 matrix and suppose that

$$\mathbf{a}_1 = 3\mathbf{a}_2 - 2\mathbf{a}_3$$

Will the system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Yes
Is A nonsingular? Explain your answers. No, by the

$a_1 - 3a_2 + 2a_3 = 0 \Rightarrow (x_1, x_2, x_3) = (1, -3, 2)$ is a solution.

If A is nonsingular then A is row equivalent to I and hence there exist elementary matrices $E_1, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_1 A = I$$

Multiplying both sides of this equation on the right by A^{-1}, we obtain

$$E_k E_{k-1} \cdots E_1 I = A^{-1}$$

Thus the same series of elementary row operations that transforms a nonsingular matrix A into I will transform I into A^{-1}. This gives us a method for computing A^{-1}.

EXAMPLE 4 Compute A^{-1} if

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$$

$[A \mid I] \xrightarrow{RREF} [I \mid A^{-1}]$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3+3R_2} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \xrightarrow[R_1-\frac{1}{2}R_3]{R_2-\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \xrightarrow[\frac{1}{6}R_3]{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

EXAMPLE 5 Solve the system

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 12 \\ -x_1 - 2x_2 \qquad &= -12 \\ 2x_1 + 2x_2 + 3x_3 &= 8 \end{aligned}$$

$$\begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix}$$

$Ax = b$

$\Longrightarrow x = A^{-1} b$

Solution

The coefficient matrix of this system is the matrix A of the last example. The solution of the system is then

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 12 \\ -12 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -\frac{8}{3} \end{pmatrix} \qquad \blacksquare$$

EXERCISES

9. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}$$

(a) Verify that

$$A^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$$

$A^{-1}A = I$

or $AA^{-1} = I$

(b) Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ for the following choices of $\mathbf{b}$.

$x = A^{-1}b$

(i) $\mathbf{b} = (1, 1, 1)^T$ **(ii)** $\mathbf{b} = (1, 2, 3)^T$

(iii) $\mathbf{b} = (-2, 1, 0)^T$

Diagonal and Triangular Matrices

An $n \times n$ matrix A is said to be *upper triangular* if $a_{ij} = 0$ for $i > j$ and *lower triangular* if $a_{ij} = 0$ for $i < j$. Also, A is said to be *triangular* if it is either upper triangular or lower triangular. For example, the 3×3 matrices

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 6 & 0 & 0 \\ 1 & 4 & 3 \end{bmatrix}$$

are both triangular. The first is upper triangular and the second is lower triangular.

A triangular matrix may have 0's on the diagonal. However, for a linear system $A\mathbf{x} = \mathbf{b}$ to be in strict triangular form, the coefficient matrix A must be upper triangular with nonzero diagonal entries.

An $n \times n$ matrix A is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$. The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are all diagonal. A diagonal matrix is both upper triangular and lower triangular.

Triangular Factorization

If an $n \times n$ matrix A can be reduced to strict upper triangular form using only row operation III, then it is possible to represent the reduction process in terms of a matrix factorization. We illustrate how this is done in the next example.

$A = L\,U$

EXAMPLE 6 Let

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} = LU$$

and let us use only row operation III to carry out the reduction process. At the first step, we subtract $\frac{1}{2}$ times the first row from the second and then we subtract twice the first row from the third.

$R_2 - \frac{1}{2}R_1$
$R_3 - 2R_1$

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

To keep track of the multiples of the first row that were subtracted, we set $l_{21} = \frac{1}{2}$ and $l_{31} = 2$. We complete the elimination process by eliminating the -9 in the (3,2) position.

$R_3 + 3R_2$

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$$

Let $l_{32} = -3$, the multiple of the second row subtracted from the third row. If we call the resulting matrix U and set

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

To see why the factorization in Example 6 works, let us view the reduction process in terms of elementary matrices. The three row operations that were applied to the matrix A can be represented in terms of multiplications by elementary matrices

$R_3 - 2R_1 \longleftrightarrow E_2$

$R_3 + 3R_2 \longleftrightarrow E_3$

$R_2 - \frac{1}{2}R_1 \longleftrightarrow E_1$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_1$

$$E_3 E_2 E_1 A = U \tag{3}$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

correspond to the row operations in the reduction process. Since each of the elementary matrices is nonsingular, we can multiply equation (3) by their inverses.

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

[We multiply in reverse order because $(E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$.] However, when the inverses are multiplied in this order, the multipliers l_{21}, l_{31}, l_{32} fill in below the diagonal in the product:

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = L$$

EXERCISES

6. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}$$

(a) Find elementary matrices E_1, E_2, E_3 such that

$$E_3 E_2 E_1 A = U$$

where U is an upper triangular matrix.

(b) Determine the inverses of E_1, E_2, E_3 and set $L = E_1^{-1} E_2^{-1} E_3^{-1}$. What type of matrix is L? Verify that $A = LU$.

Example Show that the following matrix cannot be factored directly as $\mathbf{A} = \mathbf{LU}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{bmatrix}.$$