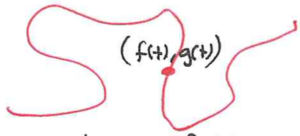
11.1 Parametrization of Plane Curves

* We may describe the movement of a particle in the xy plane at position t by $(x(t), y(t)) = (f(t), g(t))$



position of the particle at time t
not a function

Def If x and y are given as functions

$$x = f(t) \quad \text{and} \quad y = g(t) \quad , \; t \in I,$$

then the set of points $(x, y) = (f(t), g(t))$ is a parameteric curve.

Note that [1] $x = f(t)$ and $y = g(t)$ are called parametric equations.

[2] the variable t is called the parameter of the curve.

[3] the interval I is called the parameter interval.

$\Rightarrow$ If $I = [a, b]$ closed interval, then the point $(f(a), g(a))$ is the initial point and the point $(f(b), g(b))$ is the terminal point.

[4] We say that we have parametrized the curve, if we find [1] and [3]. That is [1] and [3] give a parametrization of the curve.

Exp Given the parametric equation and parameter interval:

$$x = t^2 \quad , \quad y = t + 1 \quad , \quad -\infty < t < \infty$$

[1] Find the Cartesian $\rightarrow$ algebraic equation by eliminating the parameter t

[2] Idenitify the particle's path by sketching the cartesian equation

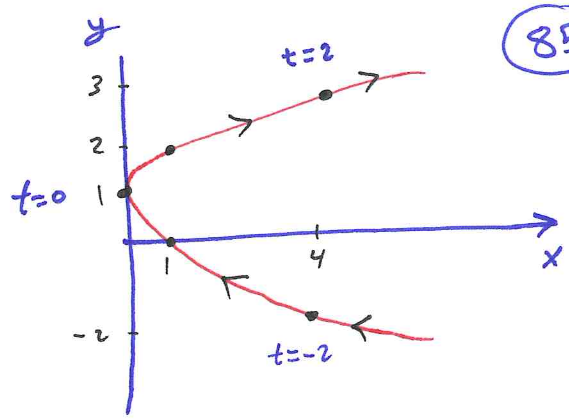
[3] Find the direction of motion

[1] Cartesian equation: $x = t^2 = (y-1)^2 \Leftrightarrow x = (y-1)^2$

Note that sometimes it's difficult or even impossible to eliminate the parameter t.

The curve that represents the particle movement.

t	-3	-2	-1	0	1	2	3
x	9	4	1	0	1	4	9
y	-2	-1	0	1	2	3	4

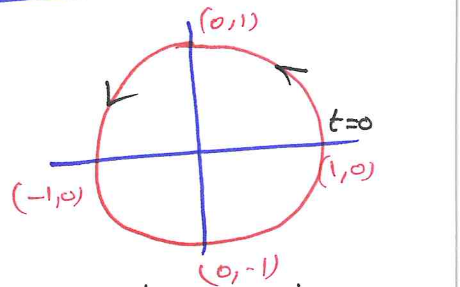


Exp Graph the parametric curve of $x = \cos t$, $y = \sin t$
$0 \le t \le 2\pi$

- we can eliminate the parameter t by:

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Leftrightarrow \boxed{x^2 + y^2 = 1}$ cartesian equation

- Initial point is $(\cos 0, \sin 0) = (1, 0)$
- Terminal point is $(\cos 2\pi, \sin 2\pi) = (1, 0)$
- $t = \pi \Rightarrow$ the position is $(-1, 0)$



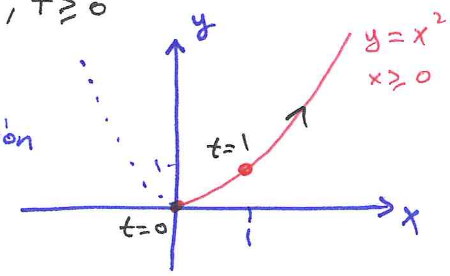
Direction: counterclockwise

Exp Graph the particle's movement and direction if its parametric equation and parameter interval is $\boxed{1}$ $x = \sqrt{t}$, $y = t$, $t \ge 0$

- We can eliminate the parameter t

$y = t = x^2 \Leftrightarrow \boxed{y = x^2}$ Cartesian equation

- Initial point is $(\sqrt{0}, 0) = (0, 0)$
- No terminal point
- $t = 1 \Rightarrow$ the position is $(1, 1)$

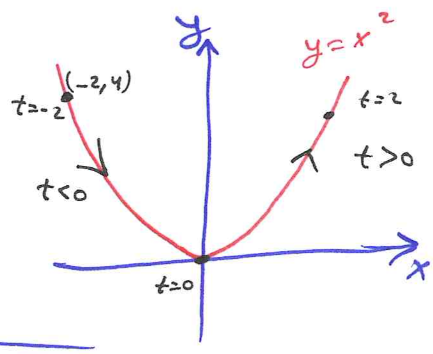


$\boxed{2}$ $x = t$, $y = t^2$, $-\infty < t < \infty$

- we can eliminate the parameter t

$y = t^2 = x^2 \Leftrightarrow \boxed{y = x^2} \rightarrow$ Cartesian equation

- no initial point
- no terminal point



Exp Find a parametrization for the line passes throw the points (a, b) and (c, d).

- A cartesian equation is $y - b = m(x-a)$ where the slope $m = \dfrac{d-b}{c-a}$, $c \neq a$
- Set the parameter $t = x - a$
- Hence, $x = a + t$, $y = b + mt$, $-\infty < t < \infty$ parameterizes the line.

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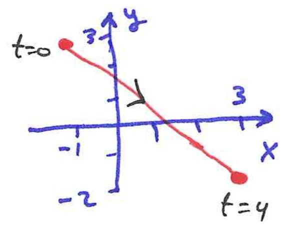
the line segment with endpoints $(-1, 3)$ and $(3, -2)$ $\quad m = \dfrac{-2-3}{3+1} = \dfrac{-5}{4}$

$$\left\{ x = -1 + t, \quad y = 3 - \frac{5}{4}t, \quad 0 \le t \le 4 \right.$$

$$\left\{ x = -1 + 4t, \quad y = 3 - 5t, \quad 0 \le t \le 1 \right\}$$



Both parametrization give the same segment

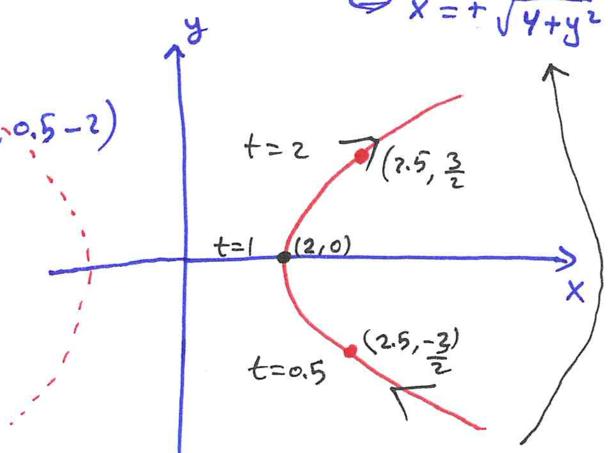
Exp sketch and identify the path by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0$$

- We can eliminate the parameter t by:

$$\left.\begin{array}{l} x + y = 2t \\ x - y = \dfrac{2}{t} \end{array}\right\} \rightarrow (x+y)(x-y) = 4 \iff x^2 - y^2 = 4 \iff x^2 = 4 + y^2 \iff x = +\sqrt{4 + y^2}$$

- at $t = 0.5 \Rightarrow$ the position is $(0.5 + 2, 0.5 - 2)$
  $\Rightarrow (2.5, \frac{-3}{2})$
- at $t = 2 \Rightarrow$ the position is $(2.5, \frac{3}{2})$



$x > 0$ since $t > 0$

Note that

$$\left\{ x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0 \right\}$$

$$\left\{ x = \sqrt{4 + t^2}, \quad y = t, \quad -\infty < t < \infty \right\}$$

$$\left\{ x = 2\sec t, \quad y = 2\tan t, \quad \frac{-\pi}{2} < t < \frac{\pi}{2} \right\}$$

are all different parametrization for the same curve.