Exercises :

2.4.0 : True or False and prove or counter example :

a. If $\{x_n\}$ is cauchy and $\{y_n\}$ is bounded then $\{x_n y_n\}$ is cauchy ? False .

$x_n = 1$ is caushy and $y_n = (-1)^n$ is bounded

But $x_n y_n = (-1)^n$ does not converge so not be cauchy .

b. If $\{x_n\}$ and $\{y_n\}$ are cauchy and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\{\frac{x_n}{y_n}\}$ is cauchy ?

False $x_n = 1$ and $b_n = \frac{1}{n}$ are cauchy .

But $\frac{x_n}{y_n} = \frac{1}{\frac{1}{n}} = n$ does not converge so not be cauchy .

c. If $\{x_n\}$ and $\{y_n\}$ are cauchy and $x_n + y_n > 0$ for all $n \in \mathbb{N}$, then $\left\{\frac{1}{x_n + y_n}\right\}$ cannot converge to zero? True

Proof: spse not, $\frac{1}{x_n + y_n} \to 0$,

$\exists K \in \mathbb{N}$ s.t $n \geq K \Rightarrow \left|\frac{1}{x_n + y_n} - 0\right| < \frac{1}{\varepsilon}$

$n \geq K \Rightarrow |x_n + y_n| > \varepsilon > 0$

In particular $|x_n + y_n|$ div to ∞ X.

contradiction → But if x_n and y_n are cauchy then $x_n + y_n \to x$ where $x \in \mathbb{R}$

Thus, $|x_n + y_n| \to |x|$ Not ∞

So $\left\{\frac{1}{x_n + y_n}\right\} \not\Rightarrow 0$

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d. If $\{x_n\}$ is sequence of real numbers that satisfies $x_{2^K} - x_{2^{K-1}} \to 0$ as $n \to \infty$ and If $x_n = 0$ for all $n \neq 2^K$, $K \in \mathbb{N}$ then $\{x_n\}$ is cauchy. False

If $x_{2^K} = \log K$ and $x_n = 0$ for $n \neq 2^K$

Then $x_{2^K} - x_{2^{K-1}} = \log\left(\frac{K}{K-1}\right) \to 0$ as $K \to \infty$ But x_K does not converges

Hence, can not be cauchy.

2.4.1 : prove that if $\{x_n\}$ is a sequence that satisfies $|x| \le \frac{2n^2+3}{n^3+5n^2+3n+1}$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is cauchy. $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t $n,m \ge N \to |x_n - x_m| < \varepsilon$.

$$\lim_{n \to \infty} \frac{2n^2+3}{n^3+5n^2+3n+1} = \lim_{n \to \infty} \frac{\frac{2}{n} + \frac{3}{n^3}}{1 + \frac{5}{n} + \frac{3}{n^2} + \frac{1}{n^3}} = 0$$

$$\to \quad 0 \le |x_n| \le \frac{2n^2+3}{n^3+5n^2+3n+1}$$

By squeez Theorem $x_n \to 0$ i.e $\{x_n\}$ is converge

So $\{x_n\}$ is cauchy.

2.4.3 : suppose that x_n and y_n are cauchy sequence in $\mathbb{R}$ and $a \in \mathbb{R}$

a. without using Thm(cauchy), prove that $a x_n$ is cauchy.

x_n is cauchy $\to \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $|x_n - x_m| < \frac{\varepsilon}{|a|}, \forall n,m \ge N$.

$$|a x_n - a x_m| = |a| \, |x_n - x_m|$$
$$< |a| \frac{\varepsilon}{|a|}$$
$$< \varepsilon \qquad \forall n,m \ge N.$$

So $a x_n$ is cauchy.

b. Without using Thm, prove $x_n + y_n$ is cauchy.

x_n is cauchy $\to$ $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $|x_n - x_m| < \frac{\varepsilon}{2}$ $\forall n, m \geq N$.

y_n is cauchy $\to$ $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $|y_n - y_m| < \frac{\varepsilon}{2}$ $\forall n, m \geq N$.

$$
\begin{aligned}
|x_n + y_n - (x_m + y_m)| &= |x_n - x_m + y_n - y_m| \\
&\leq |x_n - x_m| + |y_n - y_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon \quad \forall n, m \geq N.
\end{aligned}
$$

So $x_n + y_n$ is cauchy.

C. Without using Thm, prove that $x_n y_n$ is cauchy.

x_n is cauchy $\to$ x_n is bdd i.e $\exists M_1 \in \mathbb{R}$ s.t $|x_n| \leq M_1$.

y_n is cauchy $\to$ y_n is bdd i.e $\exists M_2 \in \mathbb{R}$ s.t $|y_n| < M_2$.

x_n is cauchy $\to$ $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $|x_n - x_m| < \frac{\varepsilon}{2M_2}$, $\forall n, m \geq N$.

y_n is cauchy $\to$ $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $|y_n - y_m| < \frac{\varepsilon}{2M_1}$, $\forall n, m \geq N$.

Now
$$
\begin{aligned}
|x_n y_n - x_m y_n| &= |x_n y_n - x_m y_n + x_m y_n - x_m y_m| \\
&\leq |x_n y_n - x_m y_n| + |x_m y_n - x_m y_m| \\
&\leq |y_n| \, |x_n - x_m| + |x_m| \, |y_n - y_m| \\
&< M_2 \frac{\varepsilon}{2M_2} + M_1 \frac{\varepsilon}{2M_1} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n, m \geq N
\end{aligned}
$$

So $x_n y_n$ is cauchy

2.4.2: suppose that $x_n \in \mathbb{Z}$ for $n \in \mathbb{N}$. If $\{x_n\}$ is cauchy prove that x_n is eventually constant; that is that there exist numbers $a \in \mathbb{Z}$ and $N \in \mathbb{N}$ s.t $x_n = a$ for all $n \geq N$.

If x_n is cauchy then $\exists N \in \mathbb{N}$ s.t $n \geq N$ implies $|x_n - x_N| < 1$.

since $x_n - x_N \in \mathbb{Z}$ it follows that $x_n = x_N \quad \forall n \geq N$

Thus, $a := x_N$.

$\boxed{\varepsilon}$

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