Control systems

Introduction to mechanical and Electromechanical modeling

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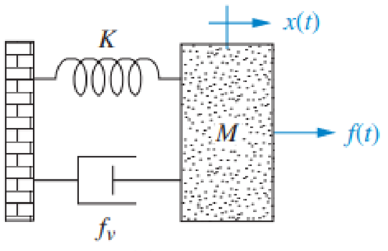
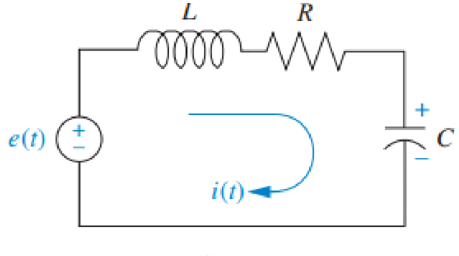
System Analogy

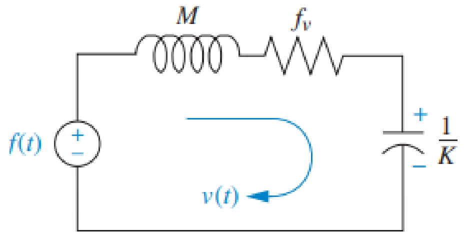
Mesh Circuit



(a)



(b)

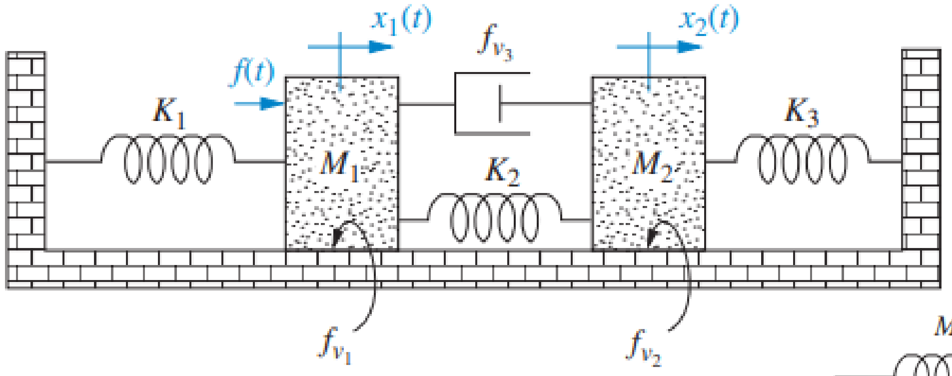


(c)

mass = M	⟶	inductor	=	M henries
viscous damper = f_v	⟶	resistor	=	f_v ohms
spring = K	⟶	capacitor	=	$\frac{1}{K}$ farads
applied force = $f(t)$	⟶	voltage source	=	$f(t)$
velocity = $v(t)$	⟶	mesh current	=	$v(t)$

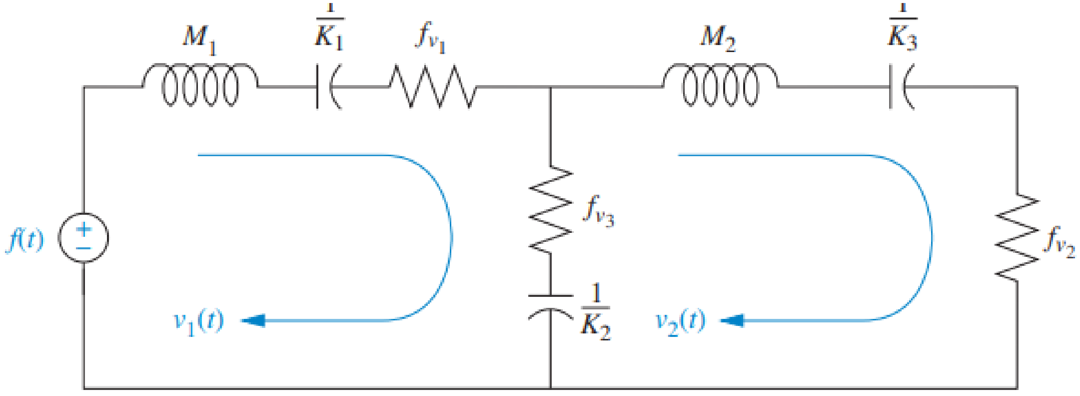
(d)

Determine the mesh analogous circuit of the following mechanical system

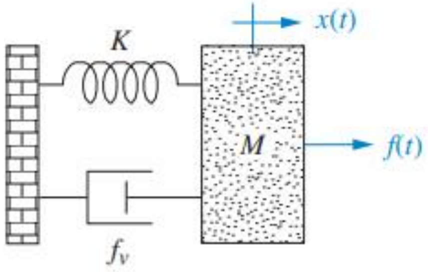


$$\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{(K_1 + K_2)}{s}\right] V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right) V_2(s) = F(s)$$

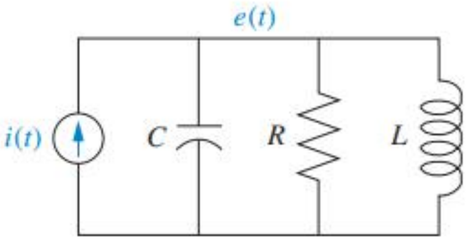
$$-\left(f_{v_3} + \frac{K_2}{s}\right) V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{(K_2 + K_3)}{s}\right] V_2(s) = 0$$



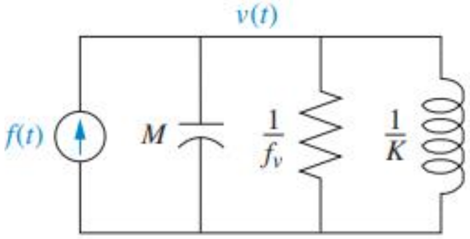
Node circuits



(a)



(b)

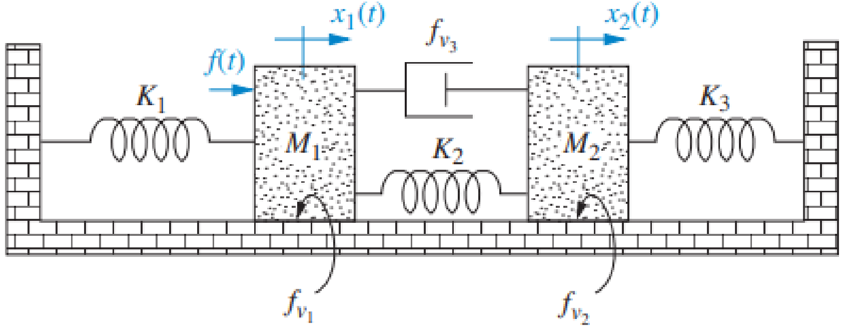


(c)

mass = M ⟶	capacitor	= M farads
viscous damper = f_v ⟶	resistor	= $\frac{1}{f_v}$ ohms
spring = K ⟶	inductor	= $\frac{1}{K}$ henries
applied force = $f(t)$ ⟶	current source	= $f(t)$
velocity = $v(t)$ ⟶	node voltage	= $v(t)$

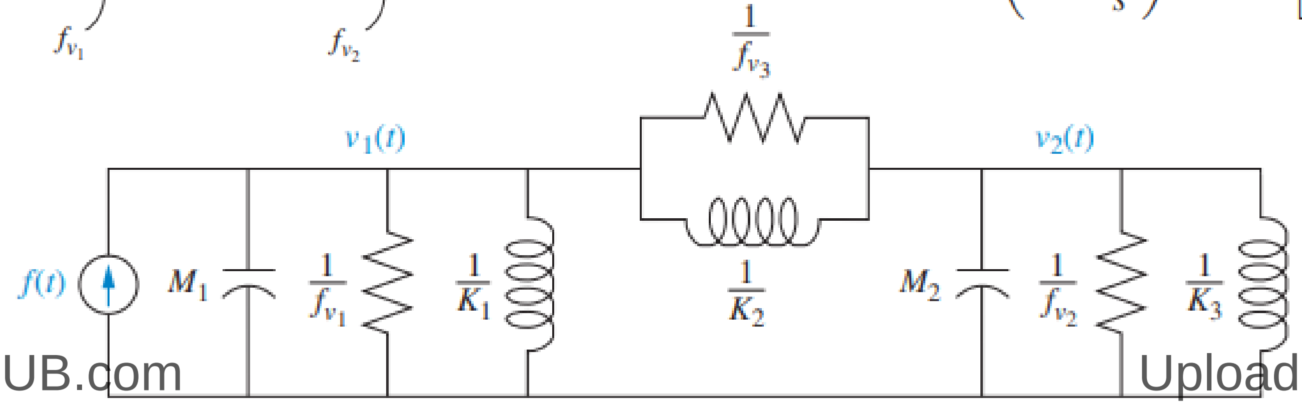
(d)

Determine the node analogous circuit of the following mechanical system



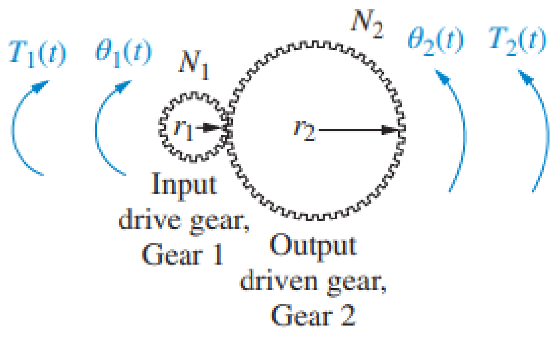
$$\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{(K_1 + K_2)}{s}\right] V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right) V_2(s) = F(s)$$

$$-\left(f_{v_3} + \frac{K_2}{s}\right) V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{(K_2 + K_3)}{s}\right] V_2(s) = 0$$



Ideal Gears:

Composed of two cylinders that transmit motion from one gear side to the other using a set of identically shaped (same pitch and height) teeth. The teeth do not slide on each other so that the circular arcs crossed by each cylinder (ignoring the teeth height) is equal. Under this condition the input and output power are equal



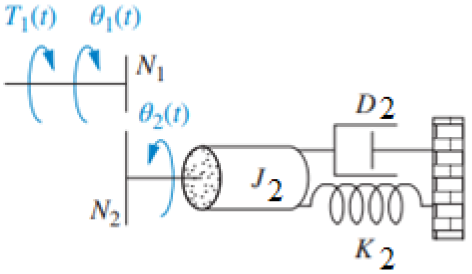
Ideal Gears motion equations:

Distance transmission $s_1 = r_1\theta_1 = r_2\theta_2 = s_2$

Energy transmission(work W) : $W_1 = T_1\theta_1 = T_2\theta_2 = W_2$

Angles-torques teeth-number relations:

The pitch angle p is equal for both cylinders→ $p = \frac{2\pi r_1}{N_1} = \frac{2\pi r_2}{N_2} \rightarrow \frac{r_1}{r_2} = \frac{N_1}{N_2} = \frac{N_1}{N_2} = \frac{\theta_2}{\theta_1} == \frac{T_1}{T_2}$



Load and external torque reflection:

Let the **source** of the reflection be side 2 and the destination **side1** and assume initially

The equation of the motion at side 2 is given by: $\boldsymbol{T_2 = (J_2 s^2 + D_2 s + K_2)\,\theta_2(s)}$

Using the angle-torque-teeth relation the equation becomes: $\boldsymbol{\frac{N_2}{N_1} T_{ref1} = (J_2 s^2 + D_2 s + K_2)\frac{N_1}{N_2}\theta_{ref1}(s)}$**, so**

$$\boldsymbol{T_{ref1} = (J_2 s^2 + D_2 s + K_2)\left(\frac{N_1}{N_2}\right)^2 \theta_{ref1}(s)}$$

<u>External torque reflection:</u> from the torque-teeth equation we can write $\boldsymbol{T_{ext-ref1}} = \frac{N_1}{N_2}\boldsymbol{T_{ext2}}$

Considering the source impedances and external torque and the fact that side1 is the destination and side 2 is the source:

$$J_{dest-tot} = J_{dest} + (\frac{N_{dest}}{N_{source}})^2\, J_{source}$$

$$D_{dest-tot} = D_{dest} + (\frac{N_{dest}}{N_{source}})^2\, D_{source}$$

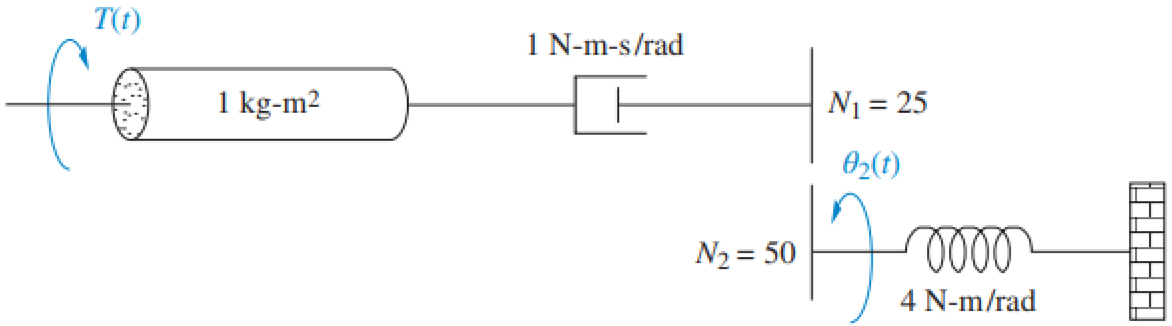
$$K_{dest-tot} = K_{dest} + (\frac{N_{dest}}{N_{source}})^2\, K_{source}$$

$$T_{ext-dest-tot} = T_{ext-dest} + (\frac{N_{dest}}{N_{source}})\, T_{ext-source}$$

The system motion equation becomes:

$$\boldsymbol{T_{ext-dest-tot} = (J_{dest-tot}s^2 + D_{dest-tot}s + K_{dest-tot})\,\theta_{dest\_tot}(s)}$$

PROBLEM: Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system with gears shown in Figure



<u>Solution:</u> we use reflection on the destination: side 1

$T_{ext-dest-tot} = T_1(s) = T(s)$, $J_{dest-tot} = J_1 + (\frac{N_1}{N_2})^2 \times 0 = 1$, $D_{dest-tot} = D_1 + (\frac{N_1}{N_2})^2 \times 0 = 1$, $K_{dest-tot} = 0 + (\frac{25}{50})^2 \times 4 = 1$

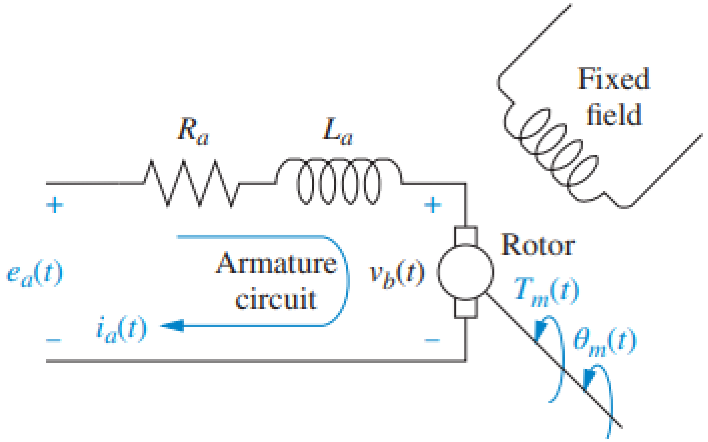
<u>Thus,</u> $T(s) = \boldsymbol{(s^2+s+1)\theta_1(s) = (s^2+s+1)\frac{N_2}{N_1}\theta_2(s) = 2(s^2+s+1)\theta_2(s) \rightarrow G(s) = \frac{\theta_2(s)}{T(s)} = \frac{1}{2(s^2+s+1)}}$

Electromechanical systems modeling: Armature controlled DC actuator:

Objective: Determine the linear time invariant model and transfer functions $G_\theta(s) = \frac{\theta_m(s)}{E_a(s)}$ and $G_\omega(s) = \frac{\omega_m(s)}{E_a(s)} = s\,G_\theta(s)$ of the DC armature controlled actuator.

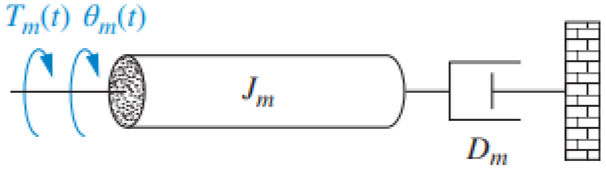
Assumptions:

- DC-Steady state model → the inductive impedance = 0.
- Linear model.
- Elastic load torques and forces =0.
- Feedback e.m.f constant $= k_b$
- Current-torque constant= k_T



Equations:

$$\begin{cases} v_b(t) = k_b \dfrac{d\theta_m(t)}{dt} = k_b\omega_m \; feedback\ electromechanical\ couplings\ \ (Farady's\ and\ Len's\ laws) \\ T(t) = k_T i_a(t) \qquad Direct\ coupling\ \ current - field\ induced\ torque \\ e_a(t) = R_a i_a(t) + v_b(t) \qquad Armature\ circuit\ mesh\ equation \\ T(t) = J_m \dfrac{d^2\theta_m(t)}{dt^2} + D_m \dfrac{d\theta_m(t)}{dt} \qquad load\ \ - torque\ mechanical\ equation \end{cases}$$



Substituting the first three equations in the forth one we obtain :

$$\frac{R_a}{k_T}\left(J_m \frac{d^2\theta_m(t)}{dt^2} + D_m \frac{d\theta_m(t)}{dt}\right) + k_b \frac{d\theta_m(t)}{dt} = e_a(t)$$

Applying Laplace transform and rearranging the equation we obtain:

$$G_\theta(s) = \frac{\frac{k_T}{R_a J_m}}{s[s+\frac{1}{J_m}\left(D_m+\frac{k_T k_b}{R_a}\right)]} \qquad G_\omega(s) = \frac{\frac{k_T}{R_a J_m}}{[s+\frac{1}{J_m}\left(D_m+\frac{k_T k_b}{R_a}\right)]}$$

Motor Torque-speed characteristic:

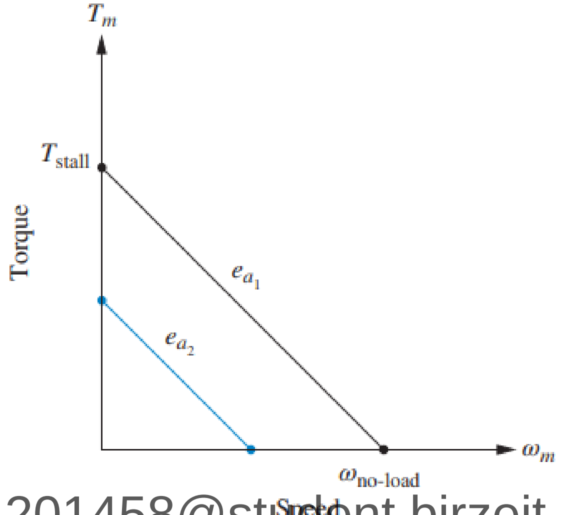
Substituting the first and second equations in the third we obtain:

$$e_a(t) = \frac{R_a}{K_T} T_m(t) + k_b \omega_m(t)$$

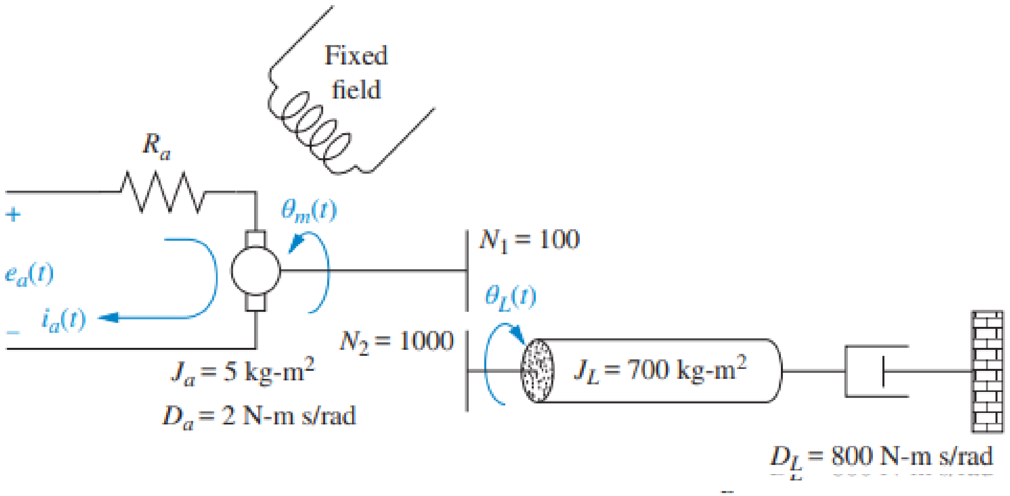
which is a family of negative slope lines with stall torque $T_{m-e_a-stall} = \frac{k_T}{R_a} e_a$ and $\omega_{m-no\_load} = \frac{1}{k_b} e_a$

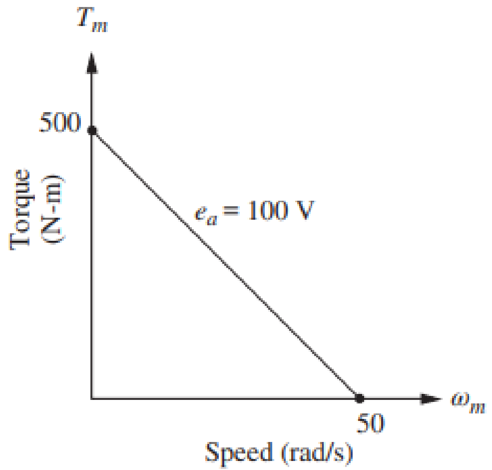
The motor parameter can be computed experimentally using

$\frac{k_T}{R_a} = \frac{T_{m-stall}}{e_a}$ and $k_b = \frac{e_a}{\omega_{m\_noload}}$



PROBLEM: Given the system and torque-speed curve of Figure , find the transfer function, $\theta_L(s)/E_a(s)$.





Solution:

We should firstly find the total mechanical impedance parameters J_m *and* D_m at the motor armature shaft and the motor characteristic parameters:

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12 \qquad D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10$$

Since We have $T_{\text{stall}} = 500 \quad \omega_{\text{no-load}} = 50 \quad e_a = 100 \quad \mapsto \quad \frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a} = \frac{500}{100} = 5 \quad K_b = \frac{e_a}{\omega_{\text{no-load}}} = \frac{100}{50} = 2$

$$\textit{The transfer function to the shaft}\; G_{\theta-shaft}(s) = \frac{\frac{k_T}{R_a J_m}}{s[s+\frac{1}{J_m}(D_m+\frac{k_T k_b}{R_a})]} = \frac{5/12}{s\left\{s+\frac{1}{12}[10+(5)(2)]\right\}} = \frac{0.417}{s(s+1.667)}$$

since $\theta_{shaft}(s) = \frac{N_{shaft}}{N_{load}}\theta_L(s) = \frac{1}{10}\theta_L(s)$ we have $G_{\theta-L}(s) = \frac{1}{10}G_{\theta-shaft}(s) = \frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)}$