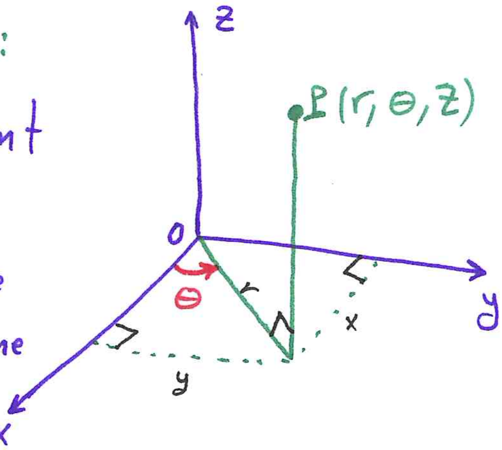
15.7 Triple Integrals in Cylindrical and spherical Coordinates:

* Integration in Cylindrical Coordinates:



Def cylindrical coordinates represent a point $P(r, \theta, z)$ in space:

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

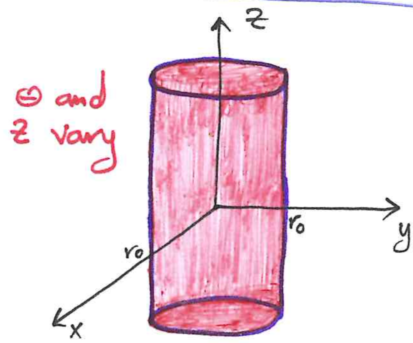
---

* Equations relating rectangular coordinate (x, y, z) and Cylindrical coordinate (r, θ, z):

$$x = r\cos\theta \;,\; y = r\sin\theta ,\; z = z \;,\; r^2 = x^2 + y^2 \;,\; \tan\theta = \frac{y}{x}$$

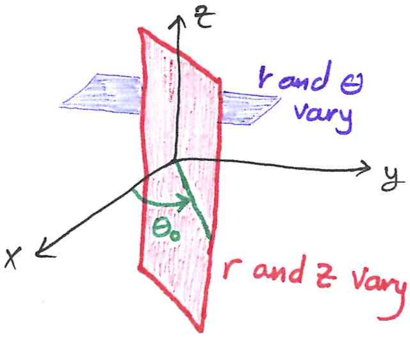
---

* In cylindrical coordinates:



➡ $r = r_0$ describes not just a circle in xy-plane but an entire cylinder about z-axis

➡ $\theta = \theta_0$ describes the plane that contains the z-axis and makes angle θ_0 with the positive x-axis.



➡ $z = z_0$ describes a plane $\perp$ z-axis

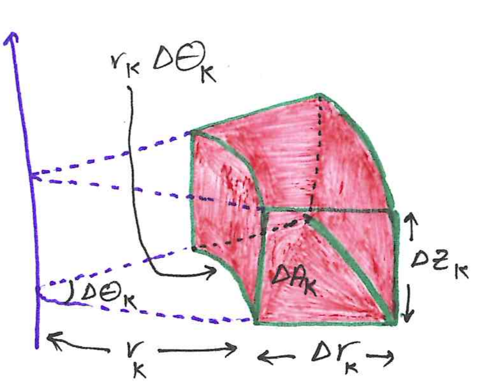
---

* To find triple integral over region D in cylindrical coordinate, we partion the region D into small n cylindrical wedges.

$\Rightarrow$ in the k^{th} cylindrical wedge: r_k, θ_k, z_k change by $\Delta r_k, \Delta\theta_k, \Delta z_k$

$\Rightarrow$ the norm of the partion $= \max\{\Delta r_k, \Delta\theta_k, \Delta z_k\}$

$\Rightarrow \Delta A_K = r_K \, \Delta r_K \, \Delta \theta_K$



$\Rightarrow \Delta V_K = \Delta z_K \, \Delta A_K$

$\Rightarrow$ For a point in the center of the K^{th} wedge, the Riemann sum of f on D has the for $S_n = \sum_{K=1}^{n} f(r_K, \theta_K, z_K) \, \Delta z_K \, r_K \, \Delta r_K \, \Delta \theta_K$

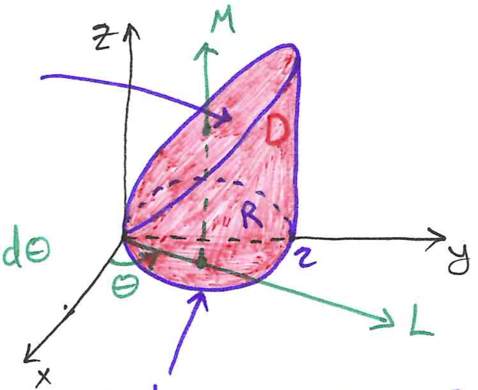
$\Rightarrow$ Take limit of S_n as $n \longrightarrow \infty$ " Norm $\to 0$ " :

$$\lim_{n\to\infty} S_n = \iiint_D f \, dv = \iiint_D f \, dz \, r dr \, d\theta$$

Exp Find the limits of integration in cylindrical coordinates for integrating the function $f(r, \theta, z)$ over D which is bounded by the plane $z=0$, the circular cylinder $x^2 + (y-1)^2 = 1$ and above by the paraboloid $z = x^2 + y^2$.

Cartesian: $z = x^2 + y^2$

cylindrical: $z = r^2$



$$\iiint_D f(r,\theta,z) \, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r,\theta,z) \, dz \, r dr \, d\theta$$

Cartesian: $x^2 + (y-1)^2 = 1$

Cylindrical: $r = 2\sin\theta$

Hence, In general:

$$\iiint_D f(r,\theta,z) \, dv = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z) \, dz \, r dr \, d\theta$$

Exp Convert the integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy$ to an equivalent integral in cylindrical coordinates and evaluate the result.

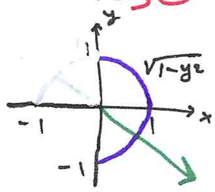
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1}\int_{0}^{r\cos\theta} r^2\,dz\,r\,dr\,d\theta$$

$x^2+y^2=r^2$

$dz\,dx\,dy \to dz\,r\,dr\,d\theta$

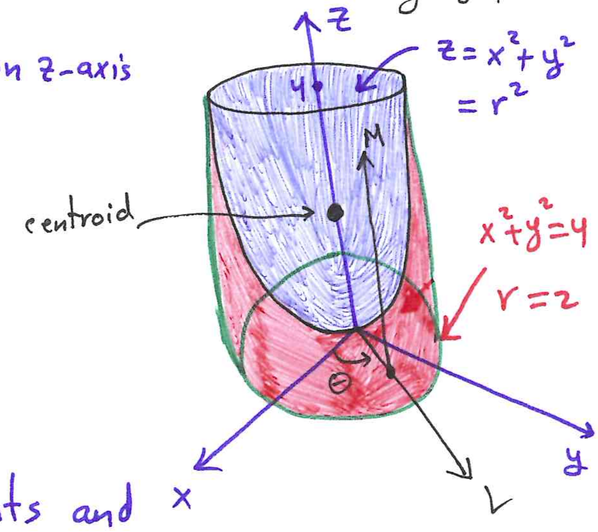
$x \to r\cos\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} r^4\cos\theta\,dr\,d\theta = \frac{1}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,d\theta = \frac{2}{5}$$



Exp Find the centroid of the solid enclosed by the cylinder $x^2+y^2=4$, bounded above by the paraboloid $z=x^2+y^2$ and below by xy-plane.

The centroid of the solid is $(\bar{x},\bar{y},\bar{z})$ on z-axis "axis of the symmetry"



- Hence, $\bar{x}=\bar{y}=0$
- Note that $\bar{x}=\dfrac{M_{yz}}{M}$, $\bar{y}=\dfrac{M_{xz}}{M}$ and

$$\bar{z}=\frac{M_{xy}}{M} \quad \text{where}$$

M_{xy}, M_{yz}, M_{xz} are the first moments and

M is the mass given by

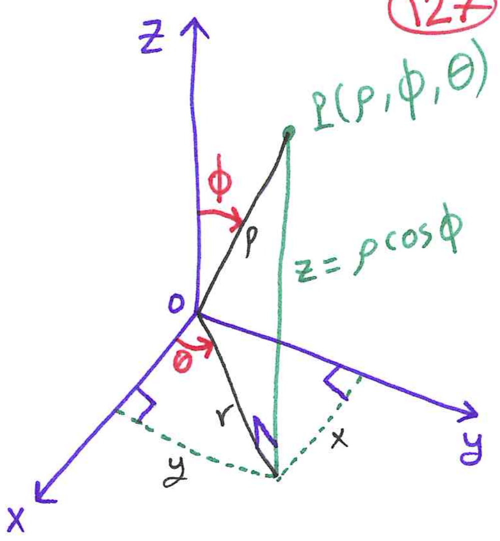
$$M=\int_{0}^{2\pi}\int_{0}^{2}\int_{0}^{r^2} dz\,r\,dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{2} r^3\,dr\,d\theta = \int_{0}^{2\pi} 4\,d\theta = 8\pi$$

$$M_{xy}=\int_{0}^{2\pi}\int_{0}^{2}\int_{0}^{r^2} z\,dz\,r\,dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{2}\frac{r^5}{2}\,dr\,d\theta = \int_{0}^{2\pi}\frac{16}{3}\,d\theta = \frac{32\pi}{3}$$

- Hence $\bar{z}=\dfrac{M_{xy}}{M}=\dfrac{32\pi}{3}\cdot\dfrac{1}{8\pi}=\dfrac{4}{3}$ and the centroid is $(0,0,\frac{4}{3})$ outside the solid.

Note: $M_{yz}=\iiint x\,dz\,r\,dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{2}\int_{0}^{r^2} r\cos\theta\,dz\,r\,dr\,d\theta = 0$ and similarly $M_{xz}=\iiint y\,dz\,r\,dr\,d\theta=0$

* Spherical Coordinates in Integral



Def Spherical coordinates represent a point $P(\rho, \phi, \theta)$ in space:

1. ρ is the distance from P to origin. That is $\rho = |\overrightarrow{OP}|$. Unlike r, $\rho \geq 0$.

2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.

3. θ is the angle from cylindrical coordinates $(0 \leq \theta \leq 2\pi)$.

---

* Equations relating spherical coordinates (ρ, ϕ, θ) to Cartesian Coordinates (x, y, z) and cylindrical Coordinates (r, θ, z):

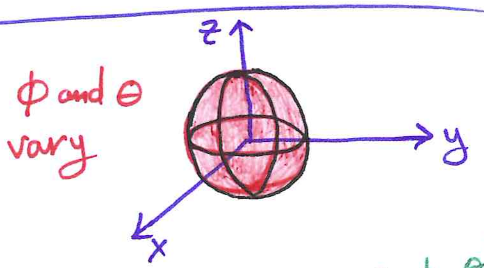
$r = \rho \sin\phi$, $z = \rho\cos\phi$, $x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

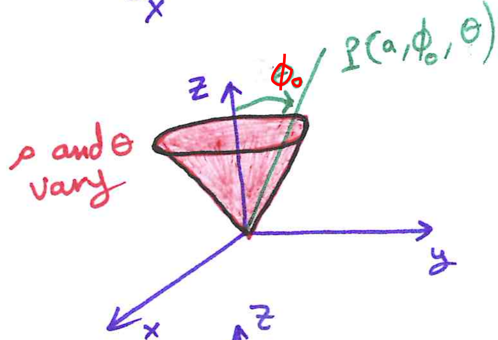
$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$$

---

* In spherical coordinates:



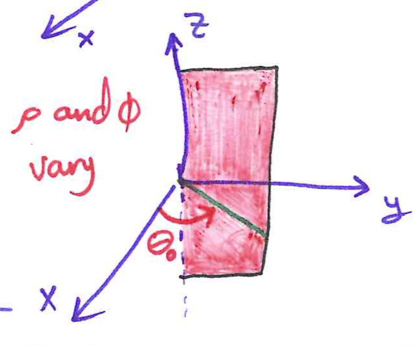
➡ $\rho = a$ describes the sphere of radius a centered at origin.



➡ $\phi = \phi_0$ describes a single cone whose vertex at origin and whose axis is the z-axis.

(As $\phi = \frac{\pi}{2} \Rightarrow$ we get xy-plane).

(The cone opens down when $\phi > \frac{\pi}{2}$).



➡ $\theta = \theta_0$ describes the half-plane that contains z-axis and make angle θ_0 with the positive x-axis.

To find triple integral over region D in spherical coordinates, we partion D into small n spherical wedges.

⇒ in the k^{th} spherical wedge : ρ_k, ϕ_k, θ_k change by $\Delta\rho_k, \Delta\phi_k, \Delta\theta_k$

⇒ $\Delta V_k = \rho_k^2 \sin\phi_k \, \Delta\rho_k \, \Delta\phi_k \, \Delta\theta_k$

⇒ The Riemann sum of f on D for a point $(\rho_k, \phi_k, \theta_k)$ inside the wedge is $S_n = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k) \, \rho_k^2 \sin\phi_k \, \Delta\rho_k \, \Delta\phi_k \, \Delta\theta_k$

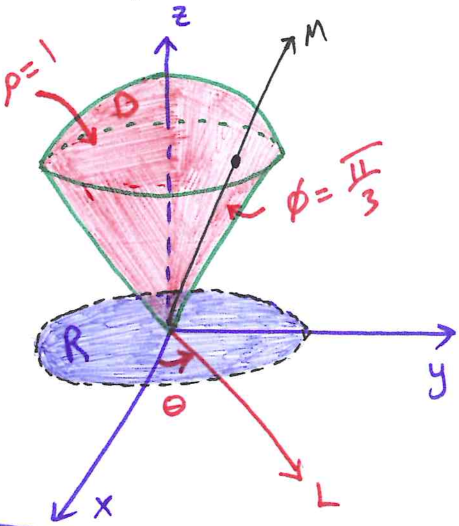
⇒ Take limit of S_n as $n \to \infty$:

$$\lim_{n\to\infty} S_n = \iiint_D f(\rho,\phi,\theta)\, dV = \iiint_D f(\rho,\phi,\theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

Exp[1] Find the volume of the ice cream cone D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \frac{\pi}{3}$

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{1} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{3}} \frac{1}{3}\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \frac{1}{6}\, d\theta = \frac{\pi}{3}$$



Hence, in general

$$\iiint_D f(\rho,\phi,\theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{\phi=\phi_{min}}^{\phi=\phi_{max}}\int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho,\phi,\theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

Exp* [2] A solid occupies the region D. Find the solid's moment of inertia about the z-axis.

In rectangular coordinate, the moment of the solid is

$$I_z = \iiint (x^2+y^2)\, dV = \iiint \underbrace{r^2}_{\rho^2 \sin^2\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \frac{1}{5}(1-\cos^2\phi)\sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{5} \int_0^{2\pi} \frac{5}{24} \, d\theta = \frac{\pi}{12}.$$

Coordinate Conversion Formulas:

[1] From Cylindrical to rectangular

$r\cos\theta \to x$

$r\sin\theta \to y$

$z \to z$

[2] From Spherical to rectangular

$\rho \sin\phi \cos\theta \to x$

$\rho \sin\phi \sin\theta \to y$

$\rho \cos\phi \to z$

[3] From Spherical to Cylindrical

$\rho \sin\phi \to r$

$\rho \cos\phi \to z$

$\theta \to \theta$

$\Rightarrow dv = dx\,dy\,dz$

$= dz\, r\,dr\, d\theta$

$= \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$