Applications of Derivatives

increasing ↑
decreasing ↓
Maximum   Max
Minimum   Min

<u>Def</u> f defined on interval I

① f ↑ on I if whenever

$$x_2 > x_1 \implies f(x_2) > f(x_1) \qquad \forall \; x_1, x_2 \in I$$

② f ↓ on I if whenever

$$x_2 > x_1 \implies f(x_2) < f(x_1) \qquad \forall \; x_1, x_2 \in I$$

<u>Exp</u> ① $f(x) = x^2$ on $[1,4]$
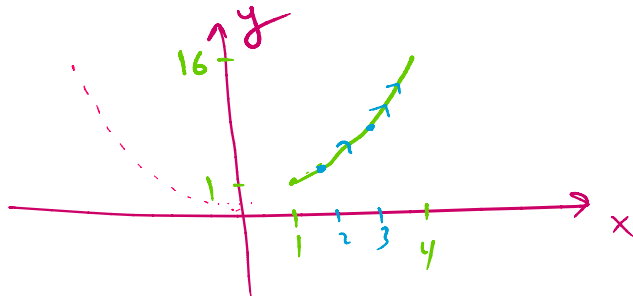
$x_1 = 2$, $x_2 = 3$

$3 > 2 \implies f(3) > f(2)$

f ↑ on $[1,4]$

② $g(x) = \dfrac{1}{x}$ on $[1,4]$
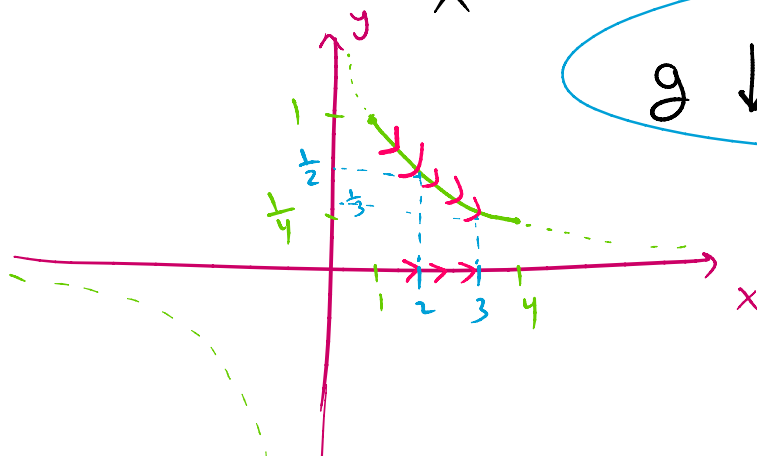
g ↓ on $[1,4]$

$x_1 = 2$, $x_2 = 3$

$3 > 2 \implies f(3) < f(2)$

$\dfrac{1}{3} < \dfrac{1}{2}$

Q. Can we use derivatives to know
f is ↑ or ↓

A. Yes
_____

Th   f cont on [a, b]
     f diff on (a, b)   Then
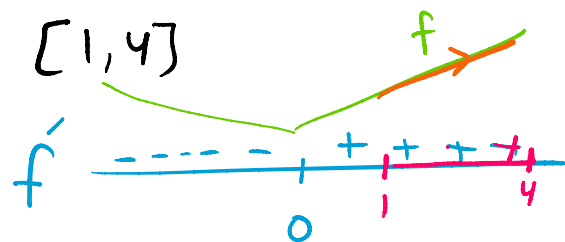
① If $f'(x) > 0$ ∀ $x \in (a,b)$ ⟹ f ↑ on [a, b]

② If $f'(x) < 0$ ∀ $x \in (a,b)$ ⟹ f ↓ on [a, b]
_____

Exp ① $f(x) = x^2$ on [1, 4]

$f'(x) = 2x$
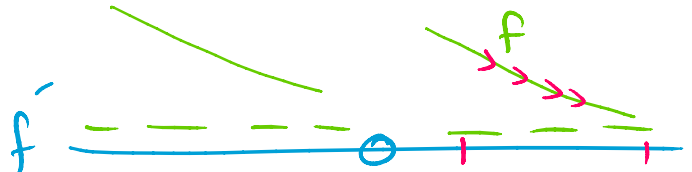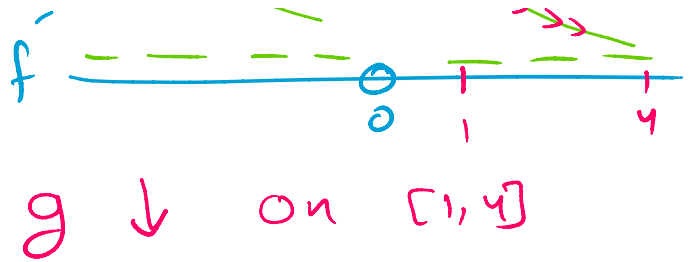
$f' = 0$ ⟹ $2x = 0$
          $x = 0$

f is ↑ on [1, 4]

② $g(x) = \dfrac{1}{x}$ on [1, 4]

$g'(x) = -\left(\dfrac{1}{x^2}\right)$

$$g'(x) = -\left(\frac{1}{x^2}\right)_{\to +}$$

f' [number line with dashes, circle at 0, marks at 1 and 4]

$$g \downarrow \quad \text{on} \quad [1, 4]$$

③ $f(x) = x^3 - 12x - 5$

$$f'(x) = 3x^2 - 12 = 0$$

$$3(x^2 - 4) = 0$$

$x^2 = 4$ ← $\boxed{x^2 - 4 = 0}$

$x = \pm 2$

$$(x-2)(x+2) = 0$$

$$x = 2, \quad x = -2$$

[graph of f]

f' [number line: +++ , dashes, +++ with marks at -2, 0, 2]
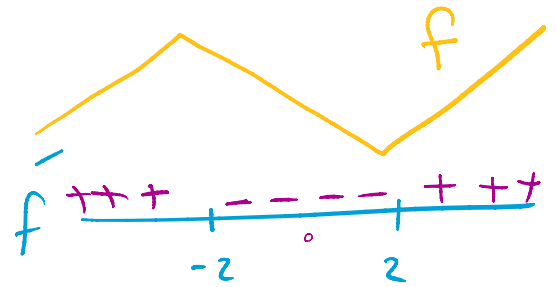
$$f'(0) = 3(0)^2 - 12$$
$$= 0 - 12$$
$$= -12 < 0$$

$$f \uparrow \quad \text{on} \quad (-\infty, -2] \cup [2, \infty)$$

$$f \downarrow \quad \text{on} \quad [-2, 2]$$

جبر صح [Arabic]

Def (Critical Point) — CP

CP's are interior points of f such that

نقاط داخلية [Arabic]

$$f'(cp) = 0 \quad \text{or} \quad f'(cp) \text{ undefined}$$

CP's $\in D(f) \to$ interior

Exp ① $f(x) = x^2$ on $[1,4]$ Find CP's

$$f'(x) = 2x = 0$$
$$x = 0 \implies 0 \notin [1,4]$$

f has no CP's on $[1,4]$

② $f(x) = x^2$ on $[-1,1]$ Find CP's

$$f'(x) = 2x = 0 \implies x = 0 \implies 0 \in [-1,1]$$

$x = 0$ is CP $\implies (0, f(0)) = (0,0)$
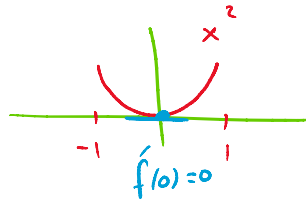
③ $g(x) = \dfrac{1}{x}$ Find CP

$$D(g) = \mathbb{R} \setminus \{0\}$$

$$g'(x) = -\frac{1}{x^2} \implies g' \text{ is undefined at } x=0$$
$$\text{but } x=0 \notin D(g)$$
$$\underline{g'(0) \text{ undefined}}$$

g has no CP's

④ $f(x) = x^3 - 12x - 5$

$D(f) = \mathbb{R} = (-\infty, \infty)$

④

$$D(f) = \mathbb{R} = (-\infty, \infty)$$

$$f' = 3x^2 - 12 = 0$$

$$x^2 - 4 = 0 \implies x = \pm 2$$

$$f'(2) = 0 \implies 2 \in D(f) \implies x = 2 \text{ is CP}$$

$$f'(-2) = 0 \implies -2 \in D(f) \implies x = -2 \text{ is CP}$$

___

Exp   Examples of ↑ functions

① $f(x) = \tan\theta$

$\boxed{f \text{ is } \uparrow}$ on domain of $\tan\theta$

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f'(x) = \boxed{\sec^2\theta} \to +$

f' $+ + + + + +$
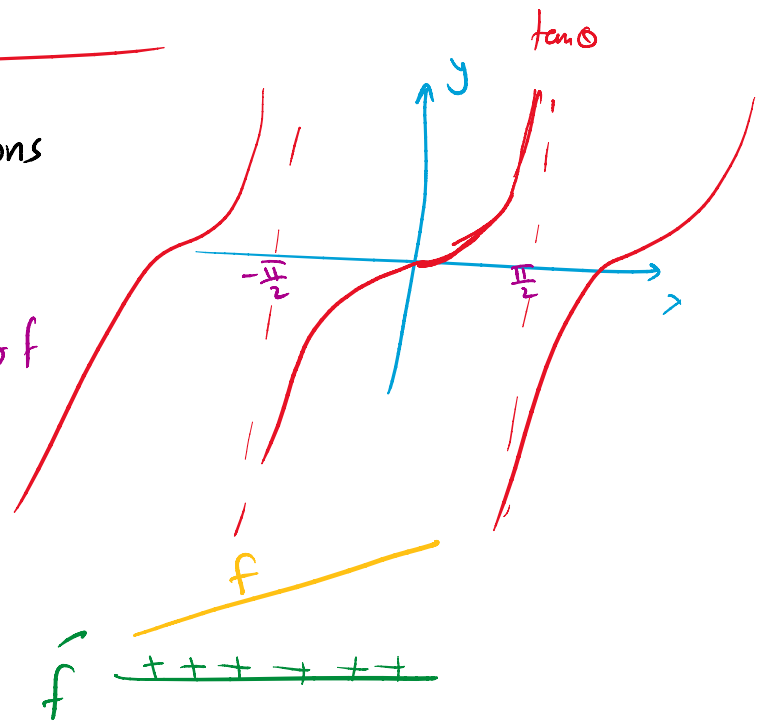
② $g(x) = x^3$ on $\mathbb{R}$

$\boxed{g \uparrow}$ on $\mathbb{R}$

$g'(x) = \boxed{3x^2} \to +$

g

g' $+ + + + + +$

$\acute{g}$ $\overline{+ + + + +}$

③ $y = e^x$

$\acute{y} = e^x$ y $\underline{+ + + +}$ $y \uparrow$ on $\mathbb{R}$

e^x
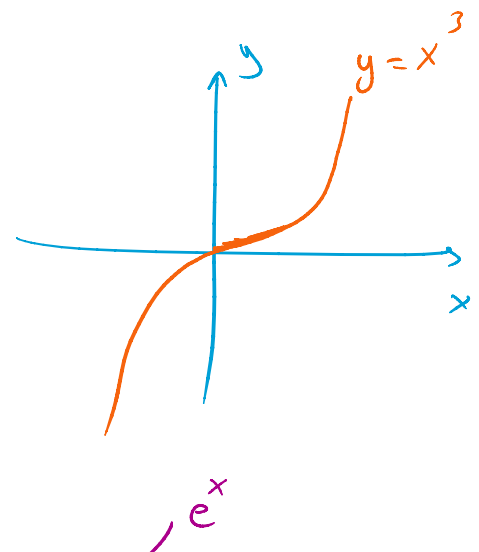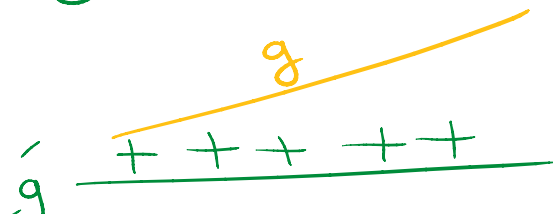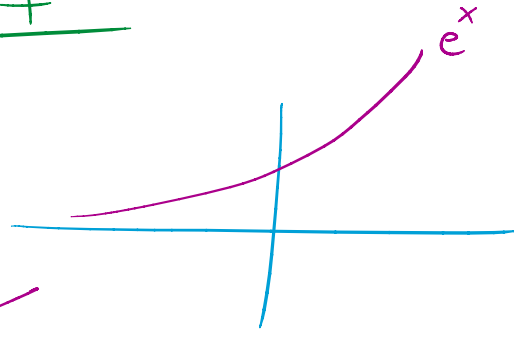
④ $y = \sqrt{x}$ on $[0, \infty)$

$\boxed{y \uparrow}$ on $[0, \infty)$

$\acute{y} = \dfrac{1}{2\sqrt{x}}$
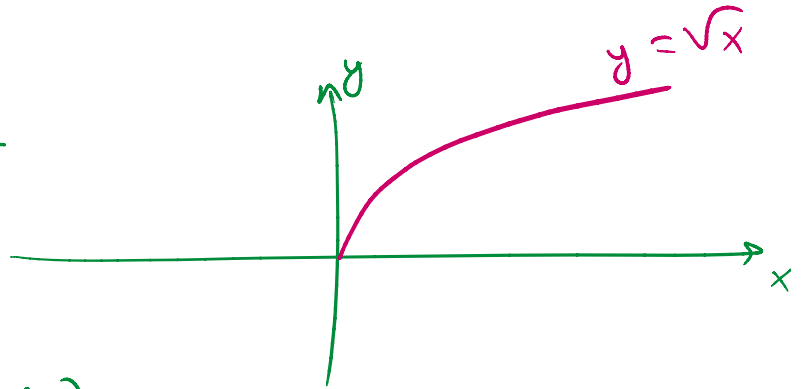
$\acute{y}_+ (0) = \infty$ undefined

$\acute{y}$ is positive on $(0, \infty)$

$\acute{y}$ $\underset{0}{\circ} \overline{+ + + + + +}$ y

$y \uparrow$ on $[0, \infty)$

$y = \sqrt{x}$

Extreme Values (EV's)
القيم القصوى

Max   Min

Max

Min

Abs.
Max المدى الحيوي

Local
Max محلي
مَنرَة مَيزة

Abs
Min مَنرَة مَيزة

Local
Min محلي
مَنرَة مَيزة
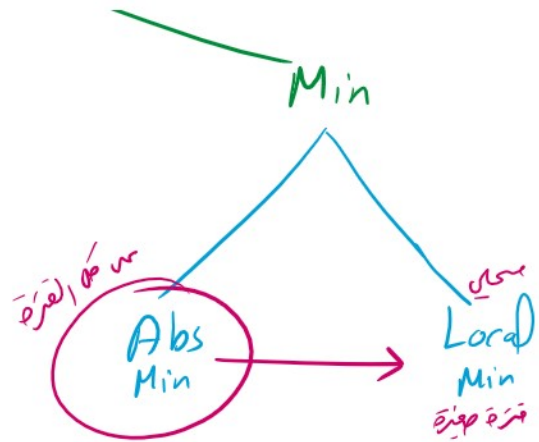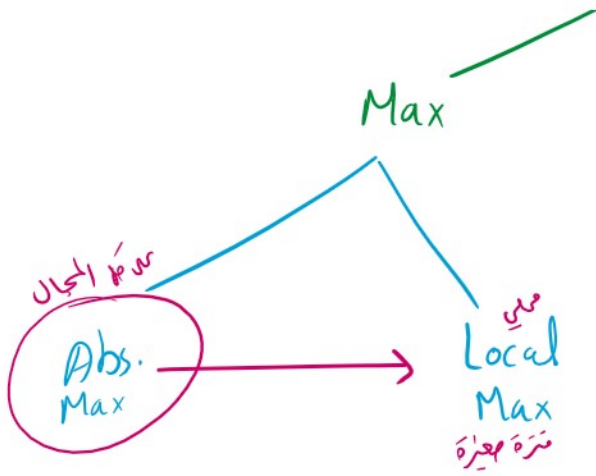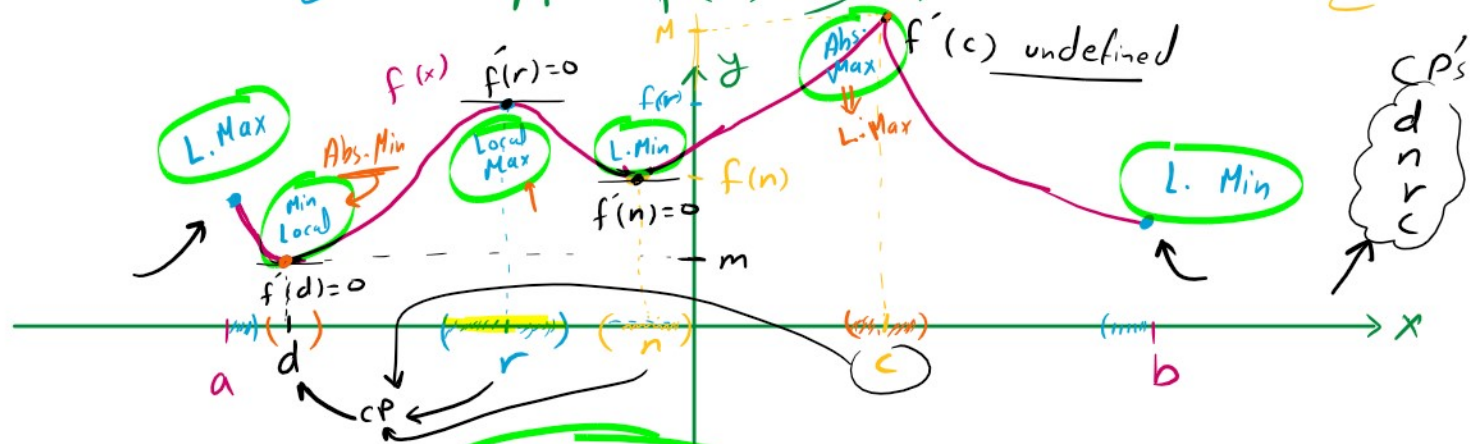
Absolutely
مطلق

Def   f   defined   domain $D = D(f)$

① f has **Abs. Max M** at point **c** $\in D$ if

$M = f(c) \geq f(x)$ $\forall x \in D$

$f'(c)$ undefined

CP's
d
n
r
c

L. Max

f(x) $f'(r) = 0$

Abs. Min

Local
Max

L. Min

$f(r)$

Abs.
Max

L. Max

L. Min

Min
Local

$f(n)$

$f'(n) = 0$

$f'(d) = 0$

M

m

a   d   cp   r   n   c   b   x

② f has **Abs. Min m** at point **d** $\in D$ if

$m = f(d) \leq f(x)$ $\forall x \in D$

③ f has **Local Max** of **f(r)** at point **r** $\in D$

(3) f has Local max of f(r) at point r=b

since $f(r) \geq f(x) \quad \forall \; x$ in small
interval about r

(4) f has local min of $f(n)$ at point n≥b

since $f(n) \leq f(x) \quad \forall \; x$ in small
interval about n

EV's occurs at ① critical points or

may

② End point

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check if ⟸ EV's للبحث عن
they occur at العظمى
① CP's or
② End points

Exp Find EV's of $f(x) = x^3$ on $[-2, 2]$

CP's → $f'(x) = 3x^2 = 0 \Rightarrow x = 0 \in D$
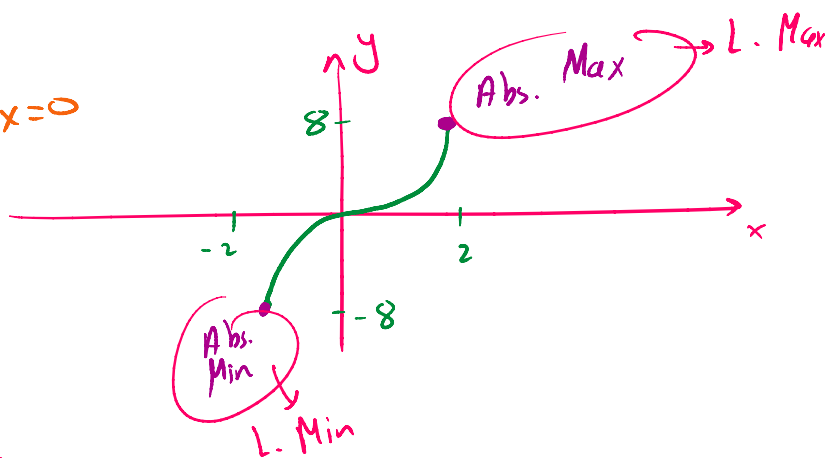
X Check at CP's
Check at Endpoints $\longrightarrow$ $f'(x) = \boxed{3x^2} = 0 \implies \boxed{x = 0} \in D$

f

f' | + + + , + + + + |
       -2   0   2

$\left.\begin{array}{l} x=0 \in D \\ f'(0) = 0 \end{array}\right\} \implies \begin{array}{l} x=0 \\ \text{is } CP \end{array}$

but f has no EV at $x=0$
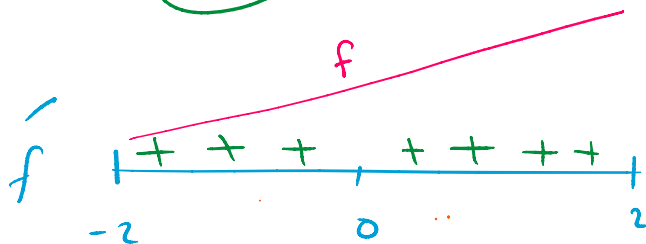
End point $f(2) = 2^3 = 8$
           $f(-2) = (-2)^3 = -8$

               L. Max
f has $\boxed{\text{Abs. Max}}$ of 8 at $x=2$
f has $\boxed{\text{Abs. Min}}$ of -8 at $x=-2$
               L. Min

Th (Extreme Value Theorem - EVT)

If f cont. on $[a,b]$

then f has Abs. Max and Abs. Min

Exp $f(x) = x^3$ on $(-2, 2)$
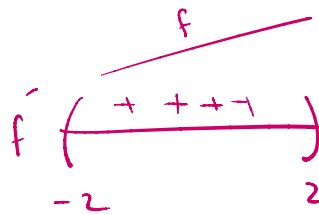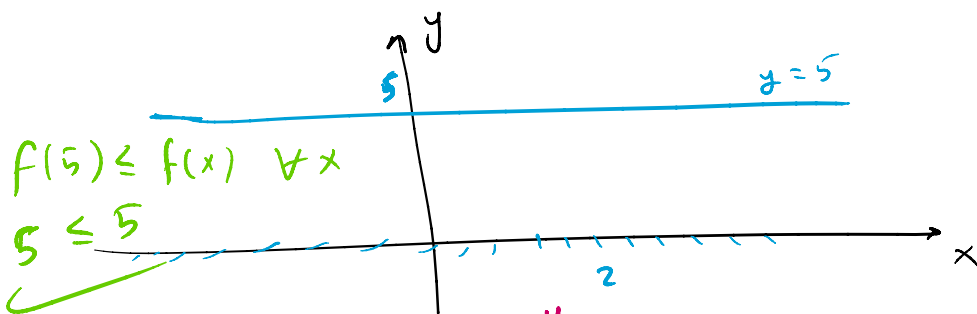
f cont. on $(-2,2)$

f has EV's

STUDENTS-HUB.com   Uploaded By: Jibreel Bornat
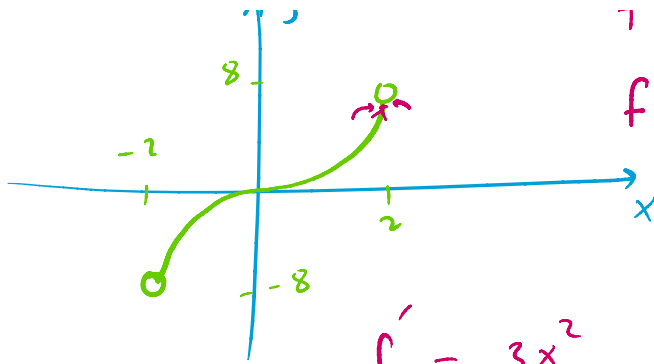
f has EV's

no

$f' = 3x^2$

$= 0$

$x = 0$

$f'\left(\overset{+\ ++\ +}{\underset{-2 \qquad\quad 2}{|\qquad\qquad|}}\right)$

Exp $f(x) = 5$ on $\mathbb{R}$

$x = 2 \implies f(2) = 5$

$f(2) \geq f(x) \ \forall x \in \mathbb{R}$

$5 \geq 5$

$y = 5$

$f(5) \leq f(x) \ \forall x$

$5 \leq 5$

L. Max

f has (Abs. Max) of 5 at $x = 5, 4, 3, 2, \ldots$

$= = = = = =$ all points in $\mathbb{R}$

f has (Abs. Min) of 5 at $x = 5, 4, 3, 2 \ldots$

$= = = = =$ all points in $\mathbb{R}$

L. Min

Exp $f(x) = \begin{cases} x & , \quad 0 \leq x < 1 \\ 0 & , \quad x = 1 \end{cases}$

Find EV's

Find EV's

CP's   End point ⟶

$$f(x) = \begin{cases} 1 & , 0 < x < 1 \\ 0 & \end{cases} \quad \boxed{x = 1}$$
↓
endpoint

$f' \neq 0$
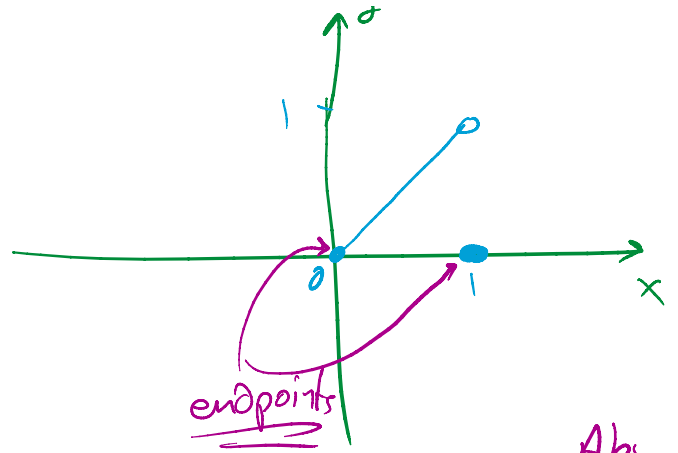
$x = 0 \Rightarrow f(0) = 0$
$x = 1 \Rightarrow f(1) = 0$ } Abs. Min ⟶
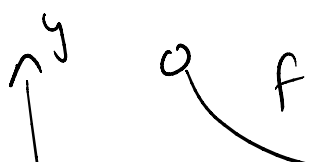
⇓

L. Min

endpoints

__Exp__
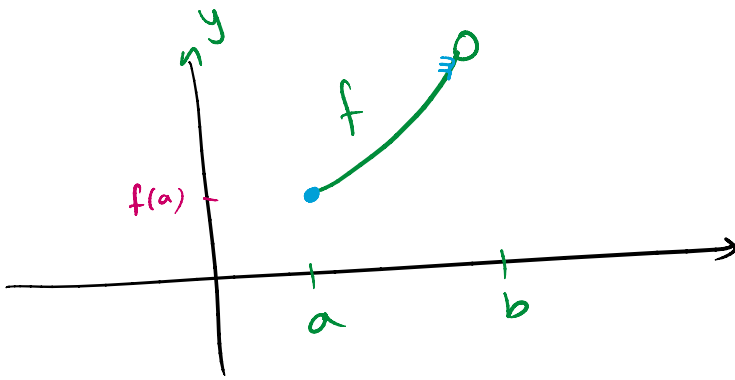
$f(a)$ is Abs. Min $\Rightarrow$
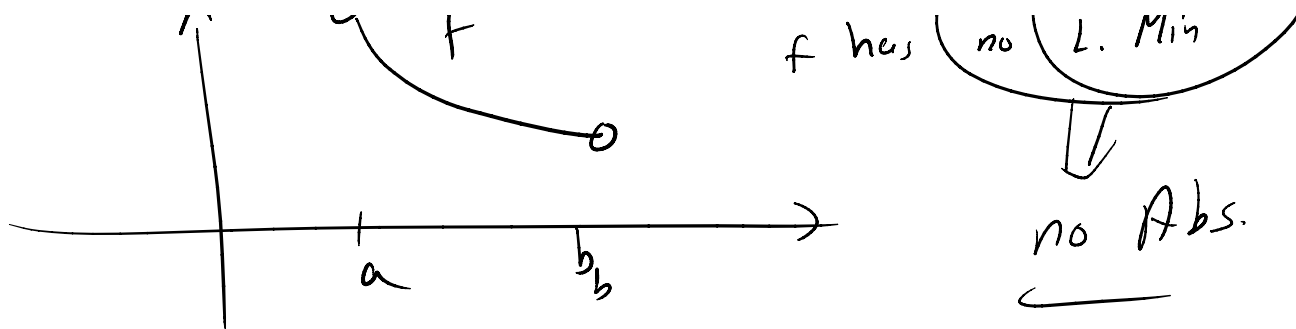$f(a)$ is L. Min

f has no Abs. Max
$\Rightarrow f = = L.$ Max
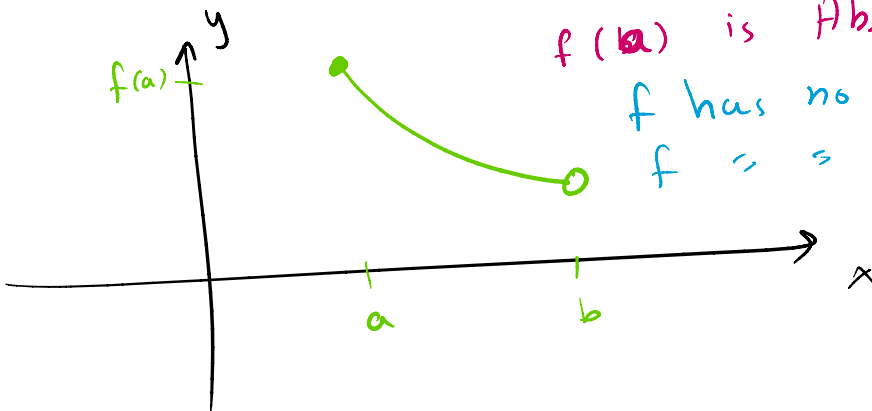
$f(a)$ is Abs. Max $\Rightarrow$ L. Max
f has no Abs. Min
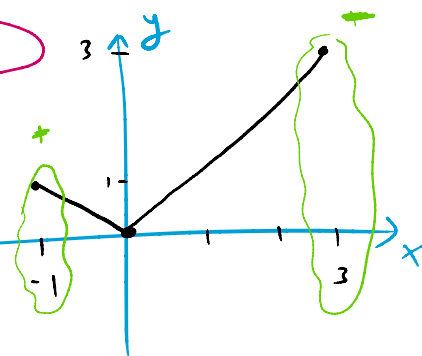$f = = L.$ Min

f has no L. Max
f has no L. Min

f has no L. Min

⟹ no Abs.

f   on   [a, b)

$f'(a)$ ⟹ $f'_+(a) = f'_-(a)$??

$\overset{+}{a}$ ————————

CP is interior ⟹ $f'(c) = 0$
                    $f'(c)$ defined

Exp $f(x) = |x|$ on $[-1, 3]$ Find EV's

⟹ f has Abs. Min of 0 at x=0

⟹ f has L. Max of 1 at x=-1

⟹ f has Abs. Max of 3 at x=3

$f(x) = |x| = \begin{cases} x & \text{if } 0 \le x \le 3 \\ -x & \text{if } -1 \le x \le 0 \end{cases}$
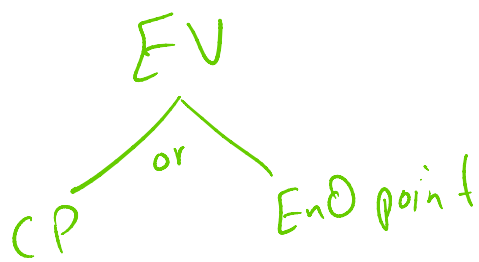
$$f(x) = |x| = \begin{cases} x & \\ -x & \text{if } -1 \le x \le 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & \text{if } 0 < x < 3 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$

$$= \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 3 \end{cases}$$

$$f'(0) \Rightarrow \emptyset \qquad f'_+(0) \ne f'_-(0)$$

$$1 \ne -$$

$$x = 0 \text{ is } \underline{\underline{CP}} \text{ and } x = 0 \in D(f)$$

EV

CP ⟍ or ⟋ End point

$$\underline{\underline{Exp}} \quad f(x) = x^{\frac{1}{3}} \text{ on } [-1, 8]$$

① Find CP's

تعريف ← or → f' undefined
f' = 0

$$f' - \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3} \frac{1}{?}$$

$$f' = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3} \frac{1}{\sqrt[3]{x^2}}$$

f' undefined at $\boxed{x = 0}$ and $0 \in [-1, 8]$

$\Rightarrow \quad x = 0 \quad$ is $\quad$ CP

② Find (EV's)

CP's $\qquad$ End point $\qquad$, $x = 8$

$\qquad\qquad\qquad\qquad x = -1 \qquad\qquad f(8) = \sqrt[3]{8}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 2$

$x = 0 \qquad\qquad\qquad\qquad f(-1) = \sqrt[3]{-1}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -1$

$f(0) = \sqrt[3]{0} = 0$

f has Abs. Min of -1 at $x = -1$

f has Abs. Max of 2 at $x = 8$

$\underline{\underline{Th}}$ f cont. on $[a, b] \Rightarrow$

f has Abs. Max and Abs. Min

$\underline{Exp}$

f cont. on (a, b)

f(c) ─
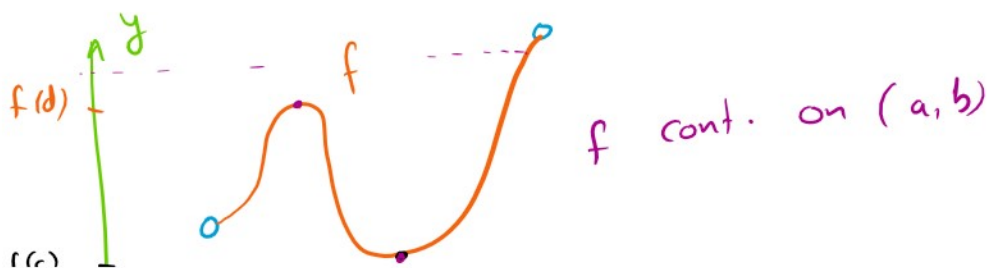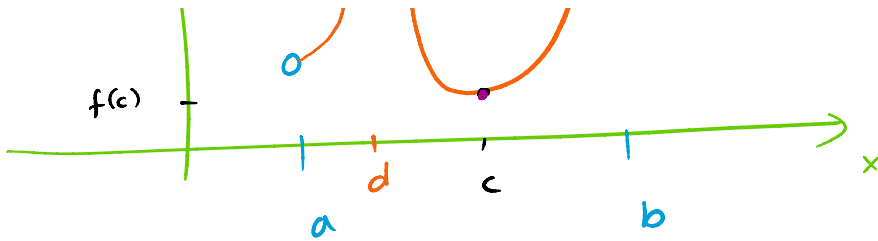
$f(c)$ is Abs. Min $\Rightarrow$ L. Min

$f(d)$ is L. Max

Abs. Max ──→ L. Max

$f(c)$

Abs. Min $f(d)$ ─

L. Min

f cont. on (a, b)