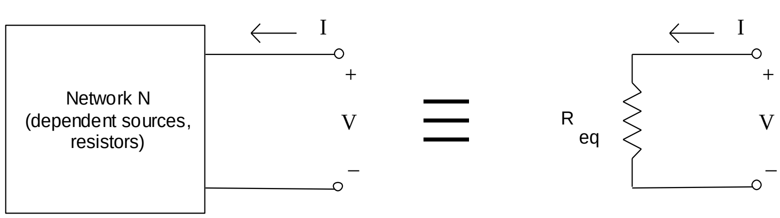
Chapter 18
Two-Port Circuits

ENEE234 – Circuit Analysis

Reading Assignment: Chapter 18 in Electric Circuits, 9th Ed. by Nilsson

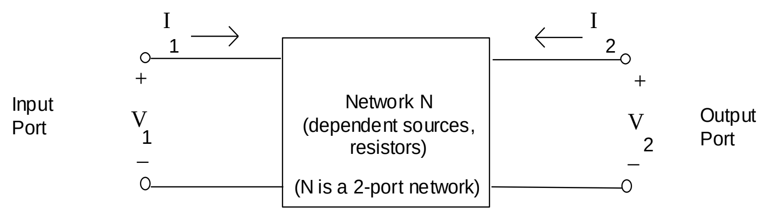
One-port networks

Earlier in the course we saw that a network N consisting or resistors and dependent sources can be represented by an equivalent resistance, R_{eq} . This type of network is a *one-port network* since it has one pair of terminals.

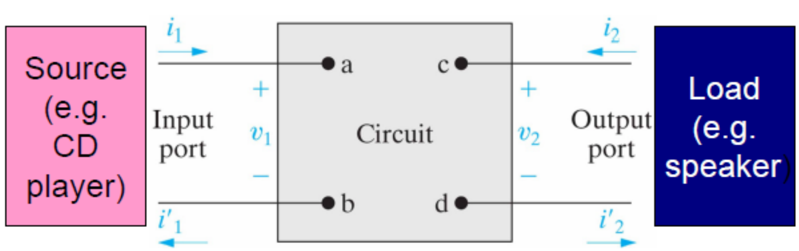


Two-port networks

Suppose that a network N has two ports as shown below. How could it be represented or modeled? A common way to represent such a network is to use one of 6 possible ***two-port networks***. These networks are circuits that are based on one of 6 possible sets of ***two-port equations***. These equations are simply different combinations of two equations that relate the variables V_1, V_2, I_1, and I_2 to one another. The coefficients in these equations are referred to as ***two-port parameters***.



How to model the “black box”?



- We will see that a two-port circuit can be modeled by a 2×2 matrix to relate the *v*/*i* variables, where the four matrix elements can be obtained by performing 2 experiments.

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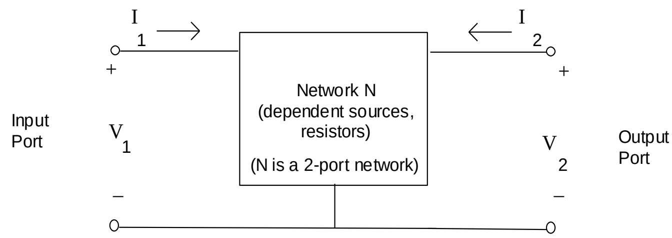
Note that I_1, I_2, V_1, and V_2 are labeled as shown by convention. Often there is a common negative terminal between the input and the output so the figure above could be redrawn as:



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Two-port equations

Two-port equations are sets of two equations relating the four variables labeled on the diagram of the two-port network above. There are 6 possible ways to form sets of two equations which express two of the variables in terms of the other two variables. The 6 possible sets of equations are shown below.

z-parameters z-parameter equations: $V_1 = z_{11} \cdot I_1 + z_{12} \cdot I_2$ $V_2 = z_{21} \cdot I_1 + z_{22} \cdot I_2$	**g-parameters** g-parameter equations: $I_1 = g_{11} \cdot V_1 + g_{12} \cdot I_2$ $V_2 = g_{21} \cdot V_1 + g_{22} \cdot I_2$
y-parameters y-parameter equations: $I_1 = y_{11} \cdot V_1 + y_{12} \cdot V_2$ $I_2 = y_{21} \cdot V_1 + y_{22} \cdot V_2$	**a-parameters**
h-parameters h-parameter equations: $V_1 = h_{11} \cdot I_1 + h_{12} \cdot V_2$ $I_2 = h_{21} \cdot I_1 + h_{22} \cdot V_2$	**b-parameters**

Applications:
z- and y-parameters: circuit modeling
y-parameters: modeling transistor capacitance at high frequencies
h- parameters: electronics (modeling transistors)

Six possible sets of terminal equations (1)

$$\begin{cases}\begin{bmatrix}V_1\\V_2\end{bmatrix}=\begin{bmatrix}z_{11} & z_{12}\\z_{21} & z_{22}\end{bmatrix}\times\begin{bmatrix}I_1\\I_2\end{bmatrix}; & [Z] \text{ is the impedance matrix;}\\ \begin{bmatrix}I_1\\I_2\end{bmatrix}=\begin{bmatrix}y_{11} & y_{12}\\y_{21} & y_{22}\end{bmatrix}\times\begin{bmatrix}V_1\\V_2\end{bmatrix}; & [Y]=[Z]^{-1} \text{ is the admittance matrix;}\end{cases}$$

$$\begin{cases}\begin{bmatrix}V_1\\I_1\end{bmatrix}=\begin{bmatrix}a_{11} & -a_{12}\\a_{21} & -a_{22}\end{bmatrix}\times\begin{bmatrix}V_2\\I_2\end{bmatrix}; & [A] \text{ is a transmission matrix;}\\ \begin{bmatrix}V_2\\I_2\end{bmatrix}=\begin{bmatrix}b_{11} & -b_{12}\\b_{21} & -b_{22}\end{bmatrix}\times\begin{bmatrix}V_1\\I_1\end{bmatrix}; & [B]=[A]^{-1} \text{ is a transmission matrix;}\end{cases}$$

Six possible sets of terminal equations (2)

$$\begin{cases}\begin{bmatrix}V_1\\I_2\end{bmatrix}=\begin{bmatrix}h_{11} & h_{12}\\h_{21} & h_{22}\end{bmatrix}\times\begin{bmatrix}I_1\\V_2\end{bmatrix}; & [H] \text{ is a hybrid matrix;}\\ \begin{bmatrix}I_1\\V_2\end{bmatrix}=\begin{bmatrix}g_{11} & g_{12}\\g_{21} & g_{22}\end{bmatrix}\times\begin{bmatrix}V_1\\I_2\end{bmatrix}; & [G]=[H]^{-1} \text{ is a hybrid matrix;}\end{cases}$$

- Which set is chosen depends on which variables are given. E.g. If the source voltage and current $\{V_1, I_1\}$ are given, choosing transmission matrix $[B]$ in the analysis.

Calculation of z-parameters

Show that

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$

.

z - parameter equations :

$$V_1 = z_{11} \cdot I_1 + z_{12} \cdot I_2$$

$$V_2 = z_{21} \cdot I_1 + z_{22} \cdot I_2$$

Similarly show expressions for the other z-parameters.

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

TABLE 18.1 Parameter Conversion Table

$$z_{11} = \frac{y_{22}}{\Delta y} = \frac{a_{11}}{a_{21}} = \frac{b_{22}}{b_{21}} = \frac{\Delta h}{h_{22}} = \frac{1}{g_{11}}$$

$$z_{12} = -\frac{y_{12}}{\Delta y} = \frac{\Delta a}{a_{21}} = \frac{1}{b_{21}} = \frac{h_{12}}{h_{22}} = -\frac{g_{12}}{g_{11}}$$

$$z_{21} = \frac{-y_{21}}{\Delta y} = \frac{1}{a_{21}} = \frac{\Delta b}{b_{21}} = -\frac{h_{21}}{h_{22}} = \frac{g_{21}}{g_{11}}$$

$$z_{22} = \frac{y_{11}}{\Delta y} = \frac{a_{22}}{a_{21}} = \frac{b_{11}}{b_{21}} = \frac{1}{h_{22}} = \frac{\Delta g}{g_{11}}$$

$$y_{11} = \frac{z_{22}}{\Delta z} = \frac{a_{22}}{a_{12}} = \frac{b_{11}}{b_{12}} = \frac{1}{h_{11}} = \frac{\Delta g}{g_{22}}$$

$$y_{12} = -\frac{z_{12}}{\Delta z} = -\frac{\Delta a}{a_{12}} = -\frac{1}{b_{12}} = -\frac{h_{12}}{h_{11}} = \frac{g_{12}}{g_{22}}$$

$$y_{21} = -\frac{z_{21}}{\Delta z} = -\frac{1}{a_{12}} = -\frac{\Delta b}{b_{12}} = \frac{h_{21}}{h_{11}} = -\frac{g_{21}}{g_{22}}$$

$$y_{22} = \frac{z_{11}}{\Delta z} = \frac{a_{11}}{a_{12}} = \frac{b_{22}}{b_{12}} = \frac{\Delta h}{h_{11}} = \frac{1}{g_{22}}$$

$$b_{21} = \frac{1}{z_{12}} = -\frac{\Delta y}{y_{12}} = \frac{a_{21}}{\Delta a} = \frac{h_{22}}{h_{12}} = -\frac{g_{11}}{g_{12}}$$

$$b_{22} = \frac{z_{11}}{z_{12}} = \frac{y_{22}}{y_{12}} = \frac{a_{11}}{\Delta a} = \frac{\Delta h}{h_{12}} = -\frac{1}{g_{12}}$$

$$h_{11} = \frac{\Delta z}{z_{22}} = \frac{1}{y_{11}} = \frac{a_{12}}{a_{22}} = \frac{b_{12}}{b_{11}} = \frac{g_{22}}{\Delta g}$$

$$h_{12} = \frac{z_{12}}{z_{22}} = -\frac{y_{12}}{y_{11}} = \frac{\Delta a}{a_{22}} = \frac{1}{b_{11}} = -\frac{g_{12}}{\Delta g}$$

$$h_{21} = -\frac{z_{21}}{z_{22}} = \frac{y_{21}}{y_{11}} = -\frac{1}{a_{22}} = -\frac{\Delta b}{b_{11}} = -\frac{g_{21}}{\Delta g}$$

$$h_{22} = \frac{1}{z_{22}} = \frac{\Delta y}{y_{11}} = \frac{a_{21}}{a_{22}} = \frac{b_{21}}{b_{11}} = \frac{g_{11}}{\Delta g}$$

$$g_{11} = \frac{1}{z_{11}} = \frac{\Delta y}{y_{22}} = \frac{a_{21}}{a_{11}} = \frac{b_{21}}{b_{22}} = \frac{h_{22}}{\Delta h}$$

$$g_{12} = -\frac{z_{12}}{z_{11}} = \frac{y_{12}}{y_{22}} = -\frac{\Delta a}{a_{11}} = -\frac{1}{b_{22}} = -\frac{h_{12}}{\Delta h}$$

$$a_{11} = \frac{z_{11}}{z_{21}} = -\frac{y_{22}}{y_{21}} = \frac{b_{22}}{\Delta b} = -\frac{\Delta h}{h_{21}} = \frac{1}{g_{21}}$$

$$a_{12} = \frac{\Delta z}{z_{21}} = -\frac{1}{y_{21}} = \frac{b_{12}}{\Delta b} = -\frac{h_{11}}{h_{21}} = \frac{g_{22}}{g_{21}}$$

$$a_{21} = \frac{1}{z_{21}} = -\frac{\Delta y}{y_{21}} = \frac{b_{21}}{\Delta b} = -\frac{h_{22}}{h_{21}} = \frac{g_{11}}{g_{21}}$$

$$a_{22} = \frac{z_{22}}{z_{21}} = -\frac{y_{11}}{y_{21}} = \frac{b_{11}}{\Delta b} = -\frac{1}{h_{21}} = \frac{\Delta g}{g_{21}}$$

$$b_{11} = \frac{z_{22}}{z_{12}} = -\frac{y_{11}}{y_{12}} = \frac{a_{22}}{\Delta a} = \frac{1}{h_{12}} = -\frac{\Delta g}{g_{12}}$$

$$b_{12} = \frac{\Delta z}{z_{12}} = -\frac{1}{y_{12}} = \frac{a_{12}}{\Delta a} = \frac{h_{11}}{h_{12}} = -\frac{g_{22}}{g_{12}}$$

$$g_{21} = \frac{z_{21}}{z_{11}} = -\frac{y_{21}}{y_{22}} = \frac{1}{a_{11}} = \frac{\Delta b}{b_{22}} = -\frac{h_{21}}{\Delta h}$$

$$g_{22} = \frac{\Delta z}{z_{11}} = \frac{1}{y_{22}} = \frac{a_{12}}{a_{11}} = \frac{b_{12}}{b_{22}} = \frac{h_{11}}{\Delta h}$$

$$\Delta z = z_{11}z_{22} - z_{12}z_{21}$$

$$\Delta y = y_{11}y_{22} - y_{12}y_{21}$$

$$\Delta a = a_{11}a_{22} - a_{12}a_{21}$$

$$\Delta b = b_{11}b_{22} - b_{12}b_{21}$$

$$\Delta h = h_{11}h_{22} - h_{12}h_{21}$$

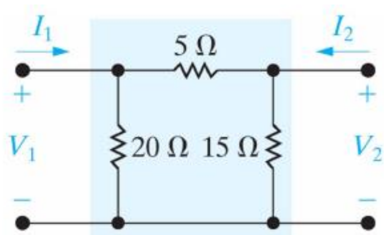
$$\Delta g = g_{11}g_{22} - g_{12}g_{21}$$

Example: Determine the z-parameters using the z-parameter definitions for the network shown below.

z - parameter equations:

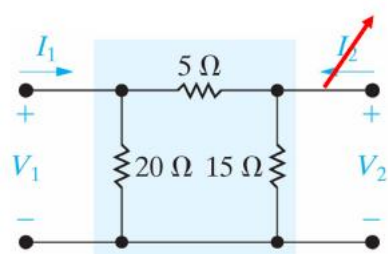
$$V_1 = z_{11} \cdot I_1 + z_{12} \cdot I_2$$

$$V_2 = z_{21} \cdot I_1 + z_{22} \cdot I_2$$



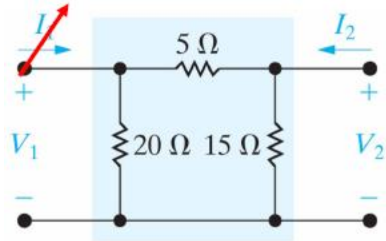
$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 20//(5+15) = 10\,\Omega$$

$$\begin{cases} V_2 = \dfrac{15\,\Omega}{5\,\Omega + 15\,\Omega} V_1 = 0.75 V_1, \\ \dfrac{V_1}{I_1} = z_{11} = 10\,\Omega, \Rightarrow I_1 = \dfrac{V_1}{10\,\Omega}, \end{cases}$$



$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{0.75 V_1}{V_1/10} = 7.5\,\Omega$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$



$$\begin{cases} V_1 = \dfrac{20\,\Omega}{20\,\Omega + 5\,\Omega} V_2 = 0.8 V_2, \\ \dfrac{V_2}{I_2} = z_{22} = 9.375\,\Omega, \Rightarrow I_2 = \dfrac{V_2}{9.375\,\Omega}, \end{cases}$$

$$\Rightarrow z_{12} = \frac{V_1}{I_2} = \frac{0.8 V_2}{V_2/(9.375\,\Omega)} = 7.5\,\Omega.$$

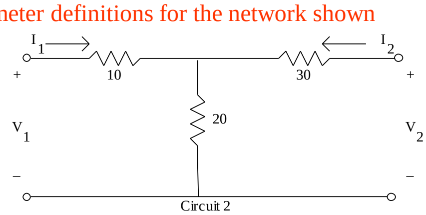
$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 15 // 25 = 9.375\,\Omega$$

Example: Determine the z-parameters using the z-parameter definitions for the network shown below.

z - parameter equations :

$V_1 = z_{11} \cdot I_1 + z_{12} \cdot I_2$

$V_2 = z_{21} \cdot I_1 + z_{22} \cdot I_2$



$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 10 + 20 = 30\,\Omega$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = 20\,\Omega$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = 20\,\Omega$$

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 30 + 20 = 50\,\Omega$$

Calculation of y-parameters

y - parameter equations :

$$I_1 = y_{11} \cdot V_1 + y_{12} \cdot V_2$$
$$I_2 = y_{21} \cdot V_1 + y_{22} \cdot V_2$$

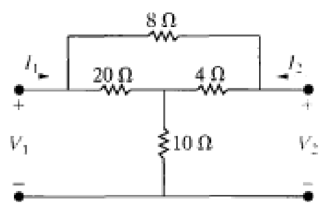
$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0}$$

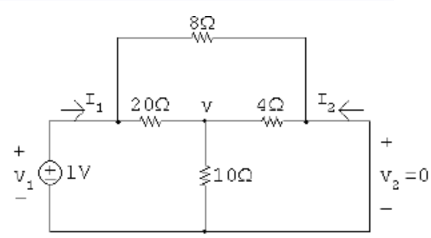
The other y-parameters.

$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1 = 0} \qquad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0}$$

$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0}$$

Calculation of y-parameters





$$\frac{V-1}{20} + \frac{V}{10} + \frac{V}{4} = 0; \quad \text{so} \quad V = 0.125\,\text{V} \quad \therefore \quad I_1 = \frac{1-0.125}{20} + \frac{1-0}{8} = 168.75\,\text{mA};$$

$$I_2 = \frac{0-0.125}{4} + \frac{0-1}{8} = -156.25\,\text{mA}$$

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0} = \frac{168.75\,\text{mA}}{1\,\text{V}} = 168.75\,\text{mS}$$

$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0} = -156.25\,\text{mS}$$

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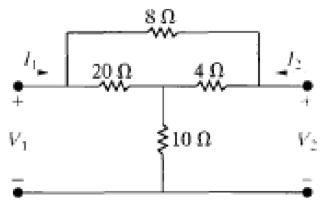
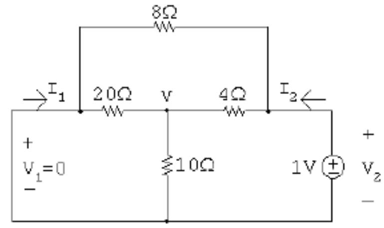
Calculation of y-parameters





$$\frac{V}{20}+\frac{V}{10}+\frac{V-1}{4}=0;\quad \text{so}\quad V=0.625\,\text{V}$$

$$\therefore\quad I_1=\frac{0-0.625}{20}+\frac{0-1}{8}=-156.25\,\text{mA};$$

$$I_2=\frac{1-0.625}{4}+\frac{1-0}{8}=218.75\,\text{mA}$$

$$y_{22}=\left.\frac{I_2}{V_2}\right|_{V_1=0}=218.75\,\text{mS}\qquad y_{12}=\left.\frac{I_1}{V_2}\right|_{V_1=0}=-156.25\,\text{mS}$$

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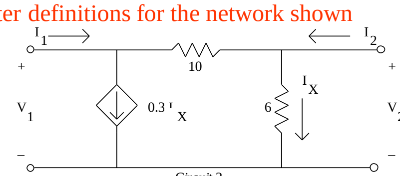
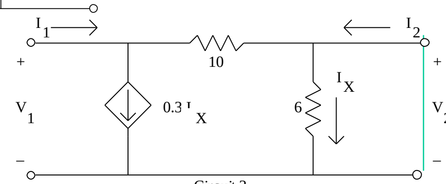
Example: Determine the y-parameters using the y-parameter definitions for the network shown below.

y - parameter equations :

$$I_1=y_{11}\cdot V_1+y_{12}\cdot V_2$$
$$I_2=y_{21}\cdot V_1+y_{22}\cdot V_2$$



Circuit 3



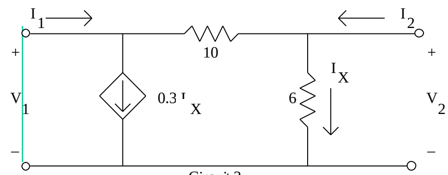
Circuit 3

$$I_x=\frac{V_2}{6}\rightarrow I_x=0$$
$$V_1=I_1*10$$

$$y_{11}=\left.\frac{I_1}{V_1}\right|_{V_2=0}=\frac{1}{10}$$

$$I_2=-I_1$$
$$V_1=-I_2*10$$

$$y_{21}=\left.\frac{I_2}{V_1}\right|_{V_2=0}=-\frac{1}{10}$$



Circuit 3

y - parameter equations:

$I_1 = y_{11} \cdot V_1 + y_{12} \cdot V_2$

$I_2 = y_{21} \cdot V_1 + y_{22} \cdot V_2$

$$V_1 = 0;$$

$$I_x = \frac{V_2}{6}$$

$$I_2 = \frac{V_2}{6} + \frac{V_2}{10}$$

$$\frac{V_2}{6} + I_1 = 0.3I_x = \frac{0.3V_2}{6}$$

$$V_2\left(\frac{1}{6} - \frac{0.3}{6}\right) = -I_1$$

$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -\frac{0.7}{6}\ \text{S}$$

$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \left(\frac{1}{10} + \frac{1}{6}\right) = 0.2667\ \text{S}$$

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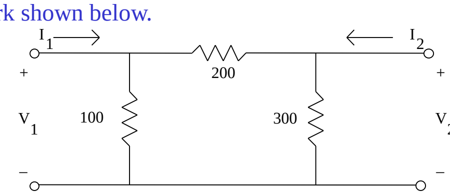
Example: Determine the h-parameters for the network shown below.



h - parameter equations:

$V_1 = h_{11} \cdot I_1 + h_{12} \cdot V_2$

$I_2 = h_{21} \cdot I_1 + h_{22} \cdot V_2$

$$V_1 = I_1(100//200)$$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = 100//200$$

$$= \frac{100 * 200}{300} = 66.7\ \Omega$$

$$\frac{V_1}{100} = I_1 + I_2$$

$$\frac{I_1 \dfrac{100 * 200}{300}}{100} = I_1 + I_2$$

$$0.667I_1 - I_1 = I_2 \rightarrow -0.333I_1 = I_2$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = -0.333$$

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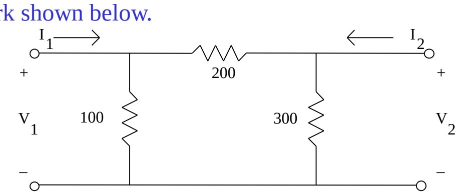
Example: Determine the h-parameters for the network shown below.



h - parameter equations:

$$V_1 = h_{11} \cdot I_1 + h_{12} \cdot V_2$$

$$I_2 = h_{21} \cdot I_1 + h_{22} \cdot V_2$$

$$I_2 = \frac{V_2}{300} + \frac{V_2 - V_1}{200}$$

$$0 = I_1 = \frac{V_1}{100} + \frac{V_1 - V_2}{200}$$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

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$$I_2 = V_2\left(\frac{1}{300} + \frac{1}{200}\right) - V_1\left(\frac{1}{100}\right) = V_2\left(\frac{1+1.5}{300}\right) - V_1\left(\frac{1}{100}\right)$$

$$= V_2\left(\frac{1+1.5}{300}\right) - V_1\left(\frac{3}{300}\right) \Longrightarrow 300I_2 = -3V_1 + 2.5V_2$$

$$0 = V_1\left(\frac{1}{100} + \frac{1}{200}\right) - V_2\left(\frac{1}{200}\right) = V_1\left(\frac{2+1}{200}\right) - V_2\left(\frac{1}{200}\right)$$

$$0 = 3V_1 - V_2$$

$$300I_2 = -3V_1 + 2.5V_2 \qquad (1)$$

$$0 = 3V_1 - V_2 \qquad (2)$$

$(1)+(2)$

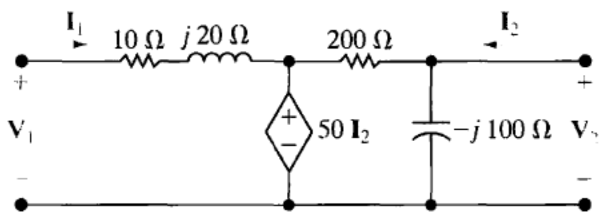
$$300I_2 = 1.5V_2$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1.5}{300} = 5\,\text{mS}$$

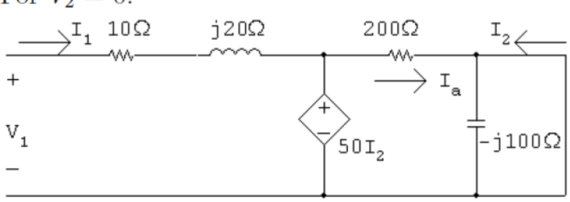
$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{1}{3}$$

18.10 Find the h parameters of the two-port circuit shown in Fig. P18.10.

Figure P18.10



For $V_2 = 0$:



$$I_a = \frac{50 I_2}{200} = \frac{1}{4} I_2 = -I_2; \qquad \therefore \qquad I_2 = 0$$

$$h_{21} = \frac{I_2}{I_1}\bigg|_{V_2=0} = 0$$

$$V_1 = (10 + j20) \qquad \therefore \qquad h_{11} = \frac{V_1}{I_1}\bigg|_{V_2=0} = 10 + j20\,\Omega$$

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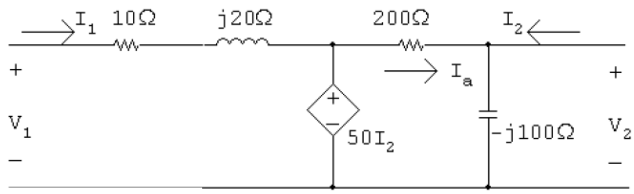
For $I_1 = 0$:



$$V_1 = 50I_2;$$

$$I_2 = \frac{V_2}{-j100} + \frac{V_2 - 50I_2}{200}$$

$$200I_2 = j2V_2 + V_2 - 50i_2$$

$$250I_2 = V_2(1 + j2)$$

$$50I_2 = V_2\left(\frac{1 + j2}{5}\right) = (0.2 + j0.4)V_2$$

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$$\therefore \quad V_1 = (0.2 + j0.4)V_2$$

$$h_{12} = \frac{V_1}{V_2}\Big|_{I_1=0} = 0.2 + j0.4$$

$$h_{22} = \frac{I_2}{V_2}\Big|_{I_1=0} = \frac{1 + j2}{250} = 4 + j8 \text{ mS}$$

Summary:

$h_{11} = 10 + j20\,\Omega;\quad h_{12} = 0.2 + j0.4;\quad h_{21} = 0;\quad h_{22} = 4 + j8\,\text{mS}$

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Calculation of 2-port parameters using network equations

Consider the z-parameter equations shown below.

$$V_1 = z_{11} \cdot I_1 + z_{12} \cdot I_2$$
$$V_2 = z_{21} \cdot I_1 + z_{22} \cdot I_2$$

Note that V_1 and V_2 are functions of I_1 and I_2 . If general sources, I_1 and I_2 are added to a network and the voltages V_1 and V_2 are calculated, the result will be expressions for V_1 and V_2 that are functions of I_1 and I_2 . So the z-parameter equations are naturally generated.

Similarly, y-parameters can be found by adding two general voltage sources V_1 and V_2 and solving for the currents I_1 and I_2 .

- Following Slides for Reference

Reciprocal Two-Port Circuits

If a two-port circuit is **reciprocal**, the following relationships exist among the port parameters:

$$z_{12} = z_{21},$$

$$y_{12} = y_{21},$$

$$a_{11}a_{22} - a_{12}a_{21} = \Delta a = 1,$$

$$b_{11}b_{22} - b_{12}b_{21} = \Delta b = 1,$$

$$h_{12} = -h_{21},$$

$$g_{12} = -g_{21}.$$

A two-port circuit is reciprocal if the interchange of an ideal voltage source at one port with an ideal ammeter at the other port produces the same ammeter reading. Consider, for example, the resistive circuit shown in Fig. 18.4. When a voltage source of 15 V is applied to port ad, it produces a current of 1.75 A in the ammeter at port cd. The ammeter current is easily determined once we know the voltage V_{bd}. Thus

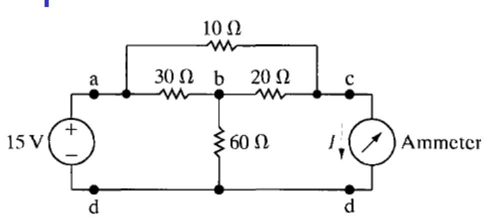

Figure 18.4 ▲ A reciprocal two-port circuit.

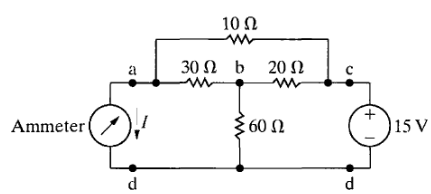

Figure 18.5 ▲ The circuit shown in Fig. 18.4, with the voltage source and ammeter interchanged.

the same voltmeter reading. For a reciprocal two-port circuit, only three calculations or measurements are needed to determine a set of parameters.

A reciprocal two-port circuit is **symmetric** if its ports can be interchanged without disturbing the values of the terminal currents and voltages. Figure 18.6 shows four examples of symmetric two-port circuits. In such circuits, the following additional relationships exist among the port parameters:

$$z_{11} = z_{22},$$

$$y_{11} = y_{22},$$

$$a_{11} = a_{22},$$

$$b_{11} = b_{22},$$

$$h_{11}h_{22} - h_{12}h_{21} = \Delta h = 1,$$

$$g_{11}g_{22} - g_{12}g_{21} = \Delta g = 1.$$

For a symmetric reciprocal network, only two calculations or measurements are necessary to determine all the two-port parameters.

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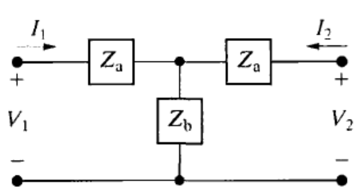
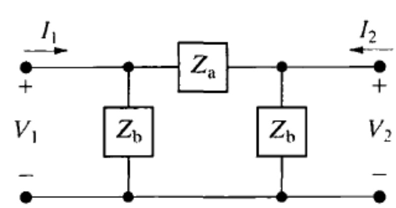
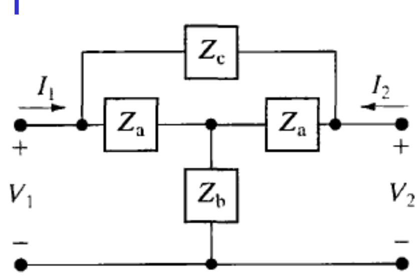


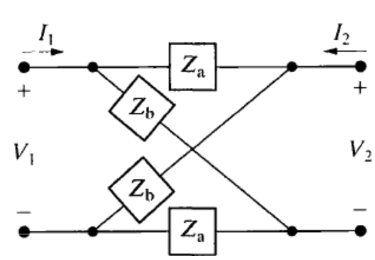
Symmetric Tee



Symmetric Pi



Symmetric Bridged Tee



Symmetric Lattice

Modeling Two-Port Networks

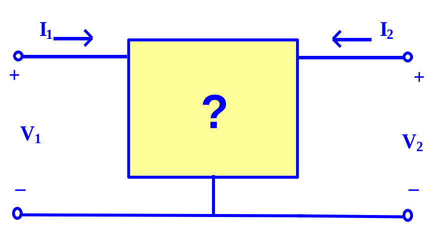
We have seen how two-port parameters can be determined for a given network. Additionally, two-port parameters might be specified for a certain device by the manufacturer (such as h-parameter values for a transistor). How are these parameters used? They are used to form a circuit model for the device or circuit. A circuit model is developed using the two-port parameter equations.

z-parameter model

The z-parameter equations are:

$$V_1 = z_{11} \cdot I_1 + z_{12} \cdot I_2$$
$$V_2 = z_{21} \cdot I_1 + z_{22} \cdot I_2$$

What sort of circuit model could be drawn that would satisfy these equations?



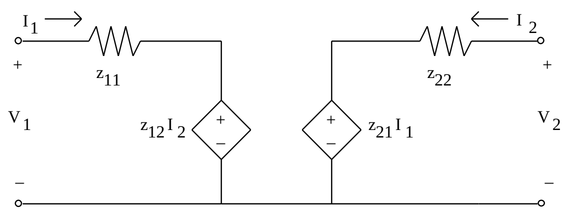
Development of the z-parameter model:

One possible circuit model could be developed by treating each of the two-port parameter equations as KVL equations (illustrate). This results in the following circuit.

z - parameter equations :

$$V_1 = z_{11} \cdot I_1 + z_{12} \cdot I_2$$

$$V_2 = z_{21} \cdot I_1 + z_{22} \cdot I_2$$



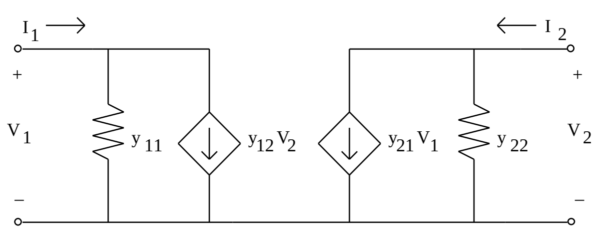
Development of the y-parameter model:

One possible circuit model could be developed by treating each of the two-port parameter equations as KCL equations (illustrate). This results in the following circuit.

y - parameter equations :

$$I_1 = y_{11} \cdot V_1 + y_{12} \cdot V_2$$

$$I_2 = y_{21} \cdot V_1 + y_{22} \cdot V_2$$



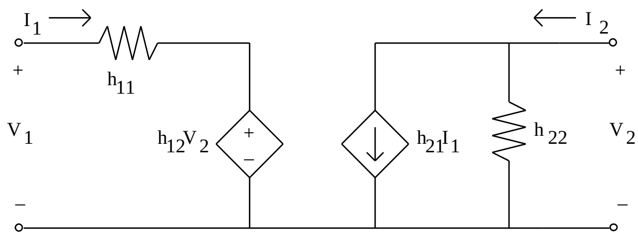
Development of the h-parameter model:

One possible circuit model could be developed by treating one of the two-port parameter equations as a KVL equation and the other as a KCL equation (illustrate). This results in the following circuit.

h - parameter equations :

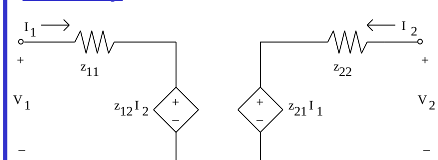
$$V_1 = h_{11} \cdot I_1 + h_{12} \cdot V_2$$

$$I_2 = h_{21} \cdot I_1 + h_{22} \cdot V_2$$



Summary:

Note: This page will be provided on the final exam.

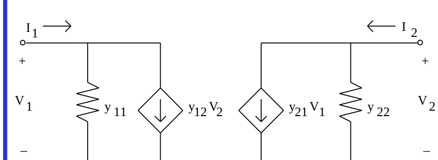


z-parameter model

z - parameter equations :

$$V_1 = z_{11} \cdot I_1 + z_{12} \cdot I_2$$

$$V_2 = z_{21} \cdot I_1 + z_{22} \cdot I_2$$

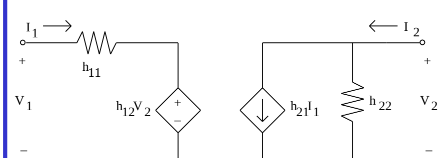


y-parameter model

y - parameter equations :

$$I_1 = y_{11} \cdot V_1 + y_{12} \cdot V_2$$

$$I_2 = y_{21} \cdot V_1 + y_{22} \cdot V_2$$



h-parameter model

h - parameter equations :

$$V_1 = h_{11} \cdot I_1 + h_{12} \cdot V_2$$

$$I_2 = h_{21} \cdot I_1 + h_{22} \cdot V_2$$

Two sets of measurements are made on a two-port resistive circuit. The first set is made with port 2 open, and the second set is made with port 2 short-circuited. The results are as follows:

Port 2 Open	Port 2 Short-Circuited
$V_1 = 10$ mV	$V_1 = 24$ mV
$I_1 = 10\ \mu$A	$I_1 = 20\ \mu$A
$V_2 = -40$ V	$I_2 = 1$ mA

Find the h parameters of the circuit.

h - parameter equations:

$V_1 = h_{11} \cdot I_1 + h_{12} \cdot V_2$

$I_2 = h_{21} \cdot I_1 + h_{22} \cdot V_2$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} \Omega, \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0},$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}, \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} \text{S.}$$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{24 \times 10^{-3}}{20 \times 10^{-6}} = 1.2\ \text{k}\Omega,$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{10^{-3}}{20 \times 10^{-6}} = 50.$$

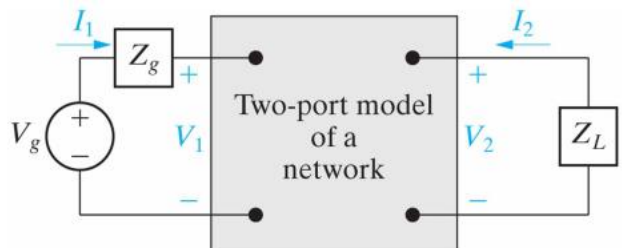
The parameters h_{12} and h_{22} cannot be obtained directly from the open-circuit test. However, a check of Eqs. 18.7–18.15 indicates that the four a parameters can be derived from the test data. Therefore, h_{12} and h_{22} can be obtained through the conversion table. Specifically,

$$h_{12} = \frac{\Delta a}{a_{22}}$$

$$h_{22} = \frac{a_{21}}{a_{22}}.$$

Model of the terminated two-port circuit

- A two-port circuit is typically driven at port 1 and loaded at port 2, which can be modeled as:



- The goal is to solve $\{V_1, I_1, V_2, I_2\}$ as functions of given parameters V_g, Z_g, Z_L, and matrix elements of the two-port circuit.

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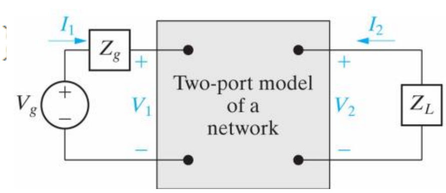
Analysis in terms of $[Z]$

- Four equations are needed to solve the four unknowns $\{V_1, I_1, V_2, I_2\}$



$$\begin{cases} V_1 = z_{11}I_1 + z_{12}I_2 \cdots (1) \\ V_2 = z_{21}I_1 + z_{22}I_2 \cdots (2) \end{cases} \cdots \text{two - port equations}$$

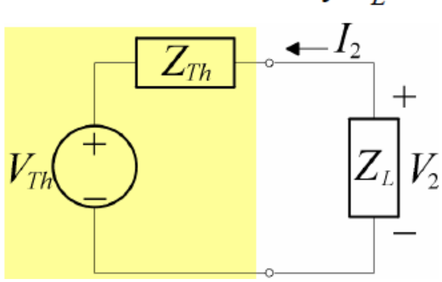
$$\begin{cases} V_1 = V_g - I_1 Z_g \cdots (3) \\ V_2 = -I_2 Z_L \cdots (4) \end{cases} \cdots \text{constraint equations due to terminations}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & z_{11} & z_{12} \\ 0 & -1 & z_{21} & z_{22} \\ 1 & 0 & Z_g & 0 \\ 0 & 1 & 0 & Z_L \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ V_g \\ 0 \end{bmatrix},$$

$\{V_1, I_1, V_2, I_2\}$ are derived by inverse matrix method.

Thévenin equivalent circuit with respect to port 2

- Once $\{V_1, I_1, V_2, I_2\}$ are solved, $\{V_{Th}, Z_{Th}\}$ can be determined by Z_L and $\{V_2, I_2\}$:



$$\begin{cases} V_2 = \dfrac{Z_L}{Z_{Th} + Z_L} V_{Th} \cdots (1) \\ I_2 = \dfrac{V_2 - V_{Th}}{Z_{Th}} \cdots (2) \end{cases}$$

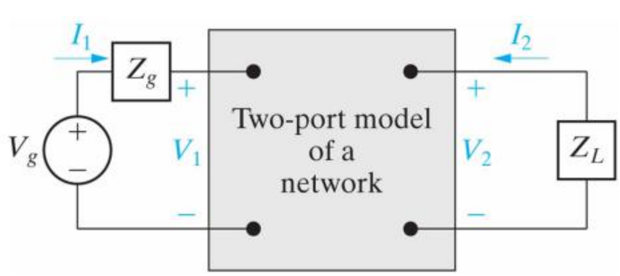
$$\Rightarrow \begin{bmatrix} Z_L & -V_2 \\ 1 & I_2 \end{bmatrix} \times \begin{bmatrix} V_{Th} \\ Z_{Th} \end{bmatrix} = \begin{bmatrix} V_2 Z_L \\ V_2 \end{bmatrix}; \quad \begin{bmatrix} V_{Th} \\ Z_{Th} \end{bmatrix} = \begin{bmatrix} Z_L & -V_2 \\ 1 & I_2 \end{bmatrix}^{-1} \times \begin{bmatrix} V_2 Z_L \\ V_2 \end{bmatrix}.$$

Terminal behavior (1)

- The terminal behavior of the circuit can be described by manipulations of $\{V_1, I_1, V_2, I_2\}$:
  - Input impedance: $Z_{in} \equiv \dfrac{V_1}{I_1} = z_{11} - \dfrac{z_{12}z_{21}}{z_{22} + Z_L};$
  - Output current: $I_2 = \dfrac{-z_{21}V_g}{(z_{11} + Z_g)(z_{22} + Z_L) - z_{12}z_{21}};$
  - Current gain: $\dfrac{I_2}{I_1} = -\dfrac{z_{21}}{z_{22} + Z_L};$
  - Voltage gains: $\begin{cases} \dfrac{V_2}{V_1} = \dfrac{z_{21}Z_L}{z_{11}Z_L + \Delta z}; \\ \dfrac{V_2}{V_g} = \dfrac{z_{21}Z_L}{(z_{11} + Z_g)(z_{22} + Z_L) - z_{12}z_{21}}; \end{cases}$

Terminal behavior (2)

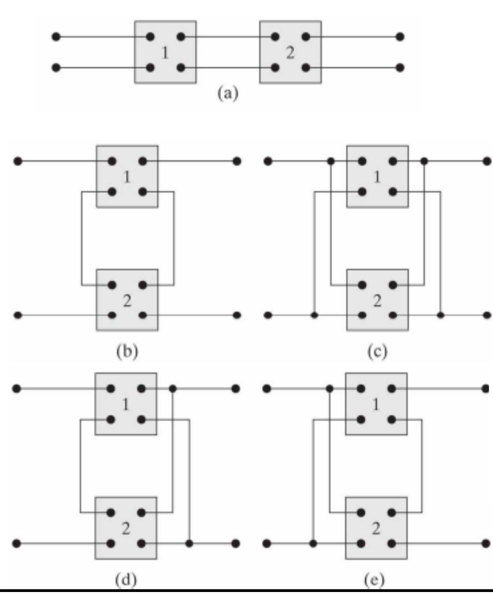
- Thévenin voltage: $V_{Th} = \dfrac{z_{21}}{z_{11} + Z_g}V_g;$
- Thévenin impedance: $Z_{Th} = z_{22} - \dfrac{z_{12}z_{21}}{z_{11} + Z_g};$



Why interconnected?

- Design of a large system is simplified by first designing subsections (usually modeled by two-port circuits), then interconnecting these units to complete the system.

Five types of interconnections of two-port circuits



a. Cascade: Better use $[A]$.

b. Series: $[Z]$

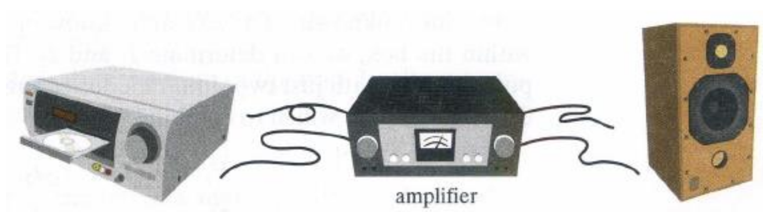
c. Parallel: $[Y]$

d. Series-parallel: $[H]$.

e. Parallel-series: $[G]$.

Practical Perspective Audio Amplifier

- Q: Whether it would be safe to use a given audio amplifier to connect a music player modeled by $\{V_g = 2\text{ V (rms)}, Z_g = 100\ \Omega\}$ to a speaker modeled by a load resistor $Z_L = 32\ \Omega$ with a power rating of 100 W?



Find the [H] by 2 test experiments (1)

- Definition of hybrid matrix [H]: $\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$;
  - Test 1:

$I_1 = 2.5\text{ mA (rms)}$

$V_1 = 1.25\text{ V (rms)}$

$V_2 = 0\text{ (short)}$

$I_2 = 3.75\text{ A (rms)}$

$$V_1 = h_{11} I_1, \Rightarrow h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{1.25\text{ V}}{2.5\text{ mA}} = 500\ \Omega.$$ Input impedance

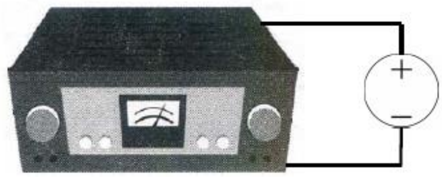
$$I_2 = h_{21} I_1, \Rightarrow h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{3.75\text{ A}}{2.5\text{ mA}} = 1500.$$ Current gain

Find the $[H]$ by 2 test experiments (2)

- Definition of hybrid matrix $[H]$: $\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$;
- Test 2:

$I_1 = 0$ (open)

$V_1 = 50$ mV (rms)



$V_2 = 50$ V (rms)

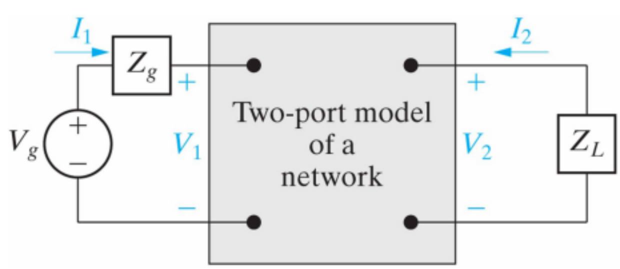
$I_2 = 2.5$ A (rms)

$$V_1 = h_{12} V_2, \Rightarrow h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{50\text{ mV}}{50\text{ V}} = 10^{-3}.$$ Voltage gain

$$I_2 = h_{22} V_2, \Rightarrow h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{2.5\text{ A}}{50\text{ V}} = (20\ \Omega)^{-1}.$$ Output admittance

Find the power dissipation on the load

- For a terminated two-port circuit:



the power dissipated on Z_L is

$$P_L = \text{Re}\{-V_2 I_2^*\} = \text{Re}\{-(-I_2 Z_L) I_2^*\} = |I_2|^2 \text{Re}\{Z_L\},$$

where I_2 is the rms output current phasor.

Method 1: Use terminated 2-port eqs for $[H]$

- By looking at Table 18.2:

$$I_2 = \frac{h_{21}V_g}{(1+h_{22}Z_L)(h_{11}+Z_g)-h_{12}h_{21}Z_L} = 1.98\text{ A (rms)},$$

$$\text{where} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 500\,\Omega & 10^{-3} \\ 1500 & (20\,\Omega)^{-1} \end{bmatrix};$$

$$V_g = 2\text{ V (rms)}, Z_g = 100\,\Omega, Z_L = 32\,\Omega.$$

Not safe!

$$\Rightarrow P_L = |I_2|^2 \operatorname{Re}\{Z_L\} = (1.98)^2(32) = 126\text{ W} > 100\text{ W}.$$