• **Differential Equation (DE)** is an equation (relation) with derivative (changes or rates)

Exp ① $y = 2x + 1$ is not DE but it is an algebraic equation with
x: independent variable
y: dependent variable

② $y' = 2x + 1$ is DE

The solution (or the unknown) is $y(x)$

③ $\frac{d^2y}{dt^2} - e^t = 3$ is DE

The solution (or the unknown) is $y(t)$

④ $\frac{dN}{dr} = N^3 \sin r$ is DE

The solution (or the unknown) is $N(r)$

Remark ① In Exp ①, ②, ③, ④:

x, t, r are indep. variables
y, N are dep. variables

② If the solution $y(x)$ passes through the point (x_0, y_0) then we write $y(x_0) = y_0$

Def The Initial Value Problem (IVP) is DE with IC where IC is Initial Condition (x_0, y_0) or $y(x_0) = y_0$

$$IVP = DE + IC$$

Exp The following are examples of IVP's:

① $\frac{dV}{dt} = 9.8 - 0.2V \quad , \quad V(\underbrace{0}_{t_0}) = \underbrace{0}_{V_0}$

The unknown (sol.) is $V(t)$

② $y'' - 2\sin x = \frac{7}{y} \quad , \quad y(\underbrace{\pi}_{x_0}) = \underbrace{1}_{y_0} \quad , \quad y'(\pi) = 2$

The unknown (sol.) is $y(x)$

③ $\frac{d^3N}{dr^3} = e^{2r} \quad , \quad N(\underbrace{e}_{r_0}) = \underbrace{2}_{N_0} \quad , \quad N'(e) = -1 \quad , \quad N''(e) = 3$

The unknown (sol.) is $N(r)$

Remark The solution of any IVP must satisfy its DE and IC.

Exp show that $y(t) = 0$ is sol. for the IVP:

$$y' = y^{\frac{1}{3}} \quad , \quad y(\underbrace{0}_{t_0}) = \underbrace{0}_{y_0}$$

- $y(t_0) = y(0) = 0 = y_0 \Rightarrow y(t) = 0$ satisfy the IC
- $y(t) = 0 \quad \left.\begin{array}{l} \Rightarrow y' = 0 \\ \Rightarrow y^{\frac{1}{3}} = 0 \end{array}\right\} \Rightarrow y' = 0 = y^{\frac{1}{3}} \Rightarrow y(t)$ satisfy the DE

Exp show that $y(x) = \frac{1}{2} e^{x^2} - \frac{1}{2}$ solves the IVP:

$$y' - 2xy - x = 0 \quad , \quad y(0) = 0$$

The sol. $y(x) = \frac{1}{2} e^{x^2} - \frac{1}{2}$ must satisfy the DE and its IC.

• IC : $y(0) = \frac{1}{2} e^{0^2} - \frac{1}{2}$

$= \frac{1}{2}(1) - \frac{1}{2}$

$= 0$ ✓

• DE : $y' = \frac{1}{2}(2x) e^{x^2} - 0 = x e^{x^2}$

$y' - 2xy - x = (x e^{x^2}) - 2x\left(\frac{1}{2} e^{x^2} - \frac{1}{2}\right) - x$

$= x e^{x^2} - x e^{x^2} + x - x$

$= 0$ ✓

Note: ① If $y = f(x)$ has all derivatives, then

Ordinary Derivatives

$y' = \frac{dy}{dx} = f'(x)$ is y prime

$y'' = \frac{d^2y}{dx^2} = f''(x)$ is y double prime

$y''' = \frac{d^3y}{dx^3} = f'''(x)$ is y tribble prime

$\vdots$

$y^{(n)} = \frac{d^ny}{dx^n} = f^{(n)}(x)$ is y super n or the n^{th} derivative of y

② If $y = f(x,s)$ has all derivatives, then

Partial Derivatives

$y_x = \frac{\partial y}{\partial x} = f_x$ | $y_{xx} = \frac{\partial^2 y}{\partial x^2} = f_{xx}$

$y_s = \frac{\partial y}{\partial s} = f_s$ | $y_{ss} = \frac{\partial^2 y}{\partial s^2} = f_{ss}$

$y_{xs} = \frac{\partial^2 y}{\partial x \partial s} = f_{xs} = f_{sx} = \frac{\partial^2 y}{\partial s \partial x}$

Exp Given the DE: $y'' + y = 0$

① Show that $y_1 = \sin t$ is sol.

We need to show $y_1'' + y_1 \overset{?}{=} 0$

$y_1' = \cos t$

$y_1'' = -\sin t$

$(-\sin t) + (\sin t) = 0$ ✓

② Verify that $y_2 = \cos t$ is sol.

We need to show that $y_2'' + y_2 \overset{?}{=} 0$

$y_2' = -\sin t$

$y_2'' = -\cos t$

$(-\cos t) + (\cos t) = 0$ ✓

③ Show that $y(t) = c_1 \sin t + c_2 \cos t$ is sol.
where $c_1, c_2 \in \mathbb{R}$

We need to show that $y'' + y \overset{?}{=} 0$

$$\left.\begin{aligned} y' &= c_1 \cos t - c_2 \sin t \\ y'' &= -c_1 \sin t - c_2 \cos t \end{aligned}\right\} \Rightarrow (-c_1 \sin t - c_2 \cos t) + (c_1 \sin t + c_2 \cos t) = 0$$ ✓

Exp ① $f(x) = 2x^2 - 4x + 1 \Rightarrow f'(x) = 4x - 4$

② $f(x,y) = 2x^2y^3 - 3e^x y + 5$ then

$f_x = \frac{\partial f}{\partial x} = 4xy^3 - 3ye^x$

$f_y = \frac{\partial f}{\partial y} = 6x^2y^2 - 3e^x$

$f_{xy} = 12xy^2 - 3e^x$
$= f_{yx}$

Def The **order** of a given DE is the highest derivative appears in the equation

Exp ① The DE $y' = y(y^2 - 3)$ has order 1 (1^{st} order)

② The DE $\frac{d^2y}{dt^2} - e^t = 3$ is of order 2

③ The DE $N''' = N^3 \sin r$ is 3^{rd} order

Question: How to classify the DE's ?

DE

ODE (Ordinary DE)

- The unknown function depends only on a single indep. variable
- Only ordinary derivatives appear in the equation
- **Exp** ① $\frac{dP}{dt} = 0.5P - 450$

  unknown $P(t)$

  ② $V'' - 2V = t$

  unknown $V(t)$
- Our Course

PDE (Partial DE)

- The unknown function depends on more than one indep. variable
- Partial derivatives appear in the equation

  $\frac{\partial^2 P}{\partial t^2} = \alpha \frac{\partial P}{\partial u}$

  unknown $P(t,u)$

  $V_t - 2V_r = tr$

  unknown $V(t,r)$
- New Course

System of DE's

$\frac{dx}{dt} = 2x - y$

$\frac{dy}{dt} = 3y - x$

Ch 7

* The general form of ODE of order n is

$$F(t, y, y', y'', \dots, y^{(n)}) = 0$$

Exp ① $y' - 2ty + 3 = 0 \Rightarrow F(t, y, y') = y' - 2ty + 3$

② $\dfrac{dN}{dx} = e^{x} - N^{2} \Rightarrow F(x, N, N') = N' - e^{x} + N^{2}$

unknown is $N(x)$ $\Rightarrow N' - e^{x} + N^{2} = 0$

③ $5y'' - 3y' = y \Rightarrow F(t, y, y', y'') = 5y'' - 3y' - y$

---

* The ODE is linear if F is linear in $y, y', y'', \dots, y^{(n)}$. Otherwise, the ODE is nonlinear.

Exp Classify the following DE's:

① $y' - 2y + 5 = 0$ $\quad 1^{st}$ order linear ODE

unknown is $y = y(x)$

② $2y'' - 5y' + 3t = 0$ $\quad 2^{nd}$ order linear ODE

unknown is $y = y(t)$

③ $\dfrac{d^6R}{dx^6} + \dfrac{d^3R}{dx^3} - 5 = e^{\sqrt{2} - x}$ $\quad 6^{th}$ order linear ODE

unknown is $R(x)$

④ $u_{xx} - u_{yy} - \cos(xy) = 0$ $\quad 2^{nd}$ order linear PDE

unknown is $u(x, y)$

⑤ $\dfrac{d^3N}{dt^3} - e^{N} \dfrac{dN}{dt} = 5t$ $\quad 3^{rd}$ order nonlinear ODE

unknown is $N(t)$

* The general form of ODE of order n is

$$F(t, y, y', y'', \dots, y^{(n)}) = 0$$

Exp ① $y' - 2ty + 3 = 0 \Rightarrow F(t, y, y') = y' - 2ty + 3$

② $\frac{dN}{dx} = e^{x} - N^{2} \Rightarrow F(x, N, N') = N' - e^{x} + N^{2}$

unknown is $N(x) \Rightarrow N' - e^{x} + N^{2} = 0$

③ $5y'' - 3y' = y \Rightarrow F(t, y, y', y'') = 5y'' - 3y' - y$

---

* The ODE is linear if F is linear in $y, y', y'', \dots, y^{(n)}$. Otherwise, the ODE is nonlinear.

Exp Classify the following DE's:

① $y' - 2y + 5 = 0$ 1^{st} order linear ODE

unknown is $y = y(x)$

② $2y'' - 5y' + 3t = 0$ 2^{nd} order linear ODE

unknown is $y = y(t)$

③ $\frac{d^6R}{dx^6} + \frac{d^3R}{dx^3} - 5 = e^{\sqrt{2} - x}$ 6^{th} order linear ODE

unknown is $R(x)$

④ $u_{xx} - u_{yy} - \cos(xy) = 0$ 2^{nd} order linear PDE

unknown is $u(x, y)$

⑤ $\frac{d^3N}{dt^3} - e^{N} \frac{dN}{dt} = 5t$ 3^{rd} order nonlinear ODE

unknown is $N(t)$

⑥ $xy' - 2y = \sin x$ $\quad 1^{st}$ order linear ODE

unknown is $y(x)$

⑦ $\frac{1}{t}\frac{dy}{dt} + (\cos t)y = t^2$ $\quad 1^{st}$ order linear ODE

unknown is $y(t)$

⑧ $(\sin t)\frac{d^2y}{dt^2} = t^3$ $\quad 2^{nd}$ order linear ODE

unknown is $y(t)$

⑨ $\left(\frac{dN}{dx}\right)^2 + N = x$ $\quad 1^{st}$ order nonlinear ODE

unknown is $N(x)$

⑩ $ty' + \frac{1}{ty} = 10$ $\quad 1^{st}$ order nonlinear ODE

unknown $y(t)$

⑪ $(x + e^y)\, dy - dx = 0$

$\Longrightarrow (x + e^y)\frac{dy}{dx} - 1 = 0$

$\frac{dy}{dx} = \frac{1}{x + e^y}$ $\quad 1^{st}$ order nonlinear ODE

unknown is $y(x)$

or $\Rightarrow (x + e^y) - \frac{dx}{dy} = 0$

$\frac{dx}{dy} = x + e^y$ $\quad 1^{st}$ order linear ODE

unknown is $x(y)$

⑫ $\frac{\partial y}{\partial x} - y\frac{\partial^2 y}{\partial x \partial s} = \sin(xs)$ $\quad 2^{nd}$ order nonlinear PDE

unknown is $y(x, s)$

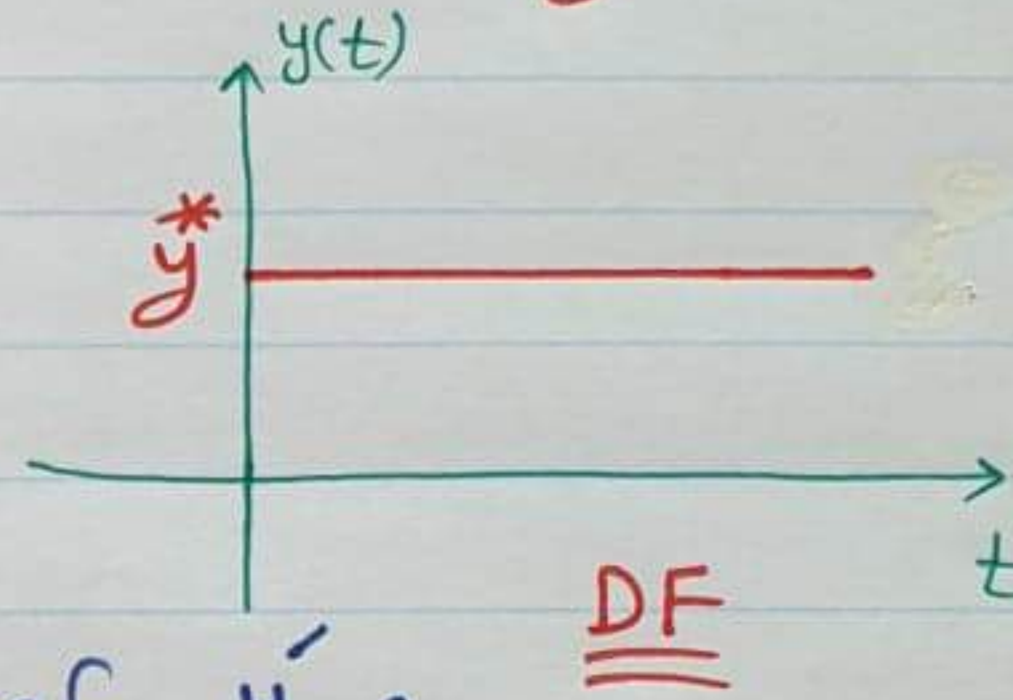
Direction Field (DF)

- We use DF to study the behaviour of the solution for a given DE without solving it.
- To draw the DF of a given DE:

$$y' = f(y) \quad \dots\dots (1)$$

→ First we find the Equilibrium Solution (Eq. Sol.) by setting $y' = 0$ and solve for y^*

→ Draw the Eq. Sol. y^*



→ substitute values of y_0 above and below y^* in (1) to see the sign of y':

→ if $y' > 0 \Rightarrow y(t) \uparrow$

→ if $y' < 0 \Rightarrow y(t) \downarrow$

→ if $y' = 0 \Rightarrow y(t) = y^*$

Exp[1] Given the DE: $y' - 2y = -4$

① Find Eq. Sol.

② Draw the DF

③ Find $\lim\limits_{t \to \infty} y(t)$ if $y_0 = 3$

① write the DE in the form (1) $\Rightarrow \boxed{y' = 2y - 4}$ (1)

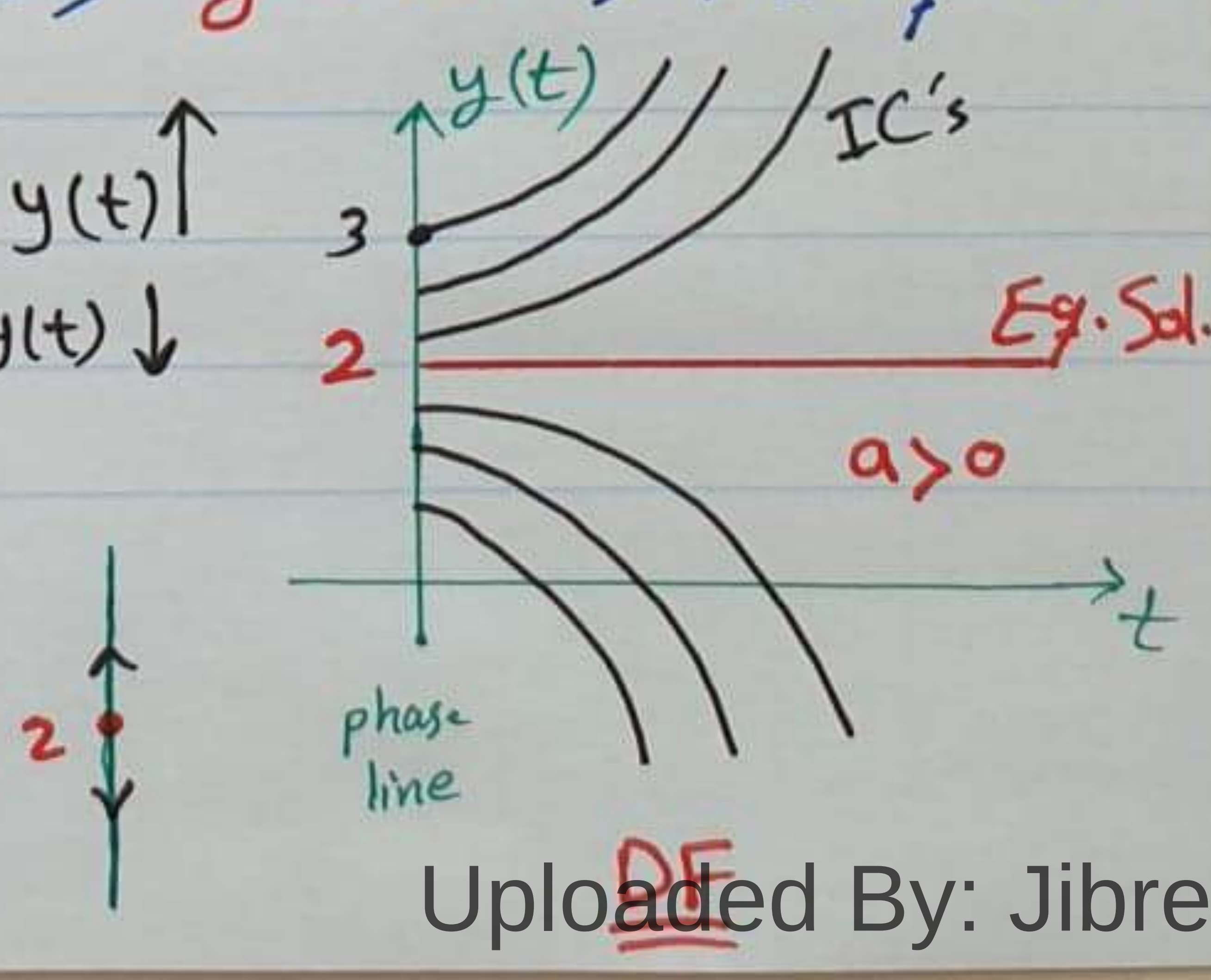
$y' = 0 \Rightarrow 2y - 4 = 0$

$\Rightarrow 2y = 4 \Rightarrow y^* = 2$ is the Eq. Sol.

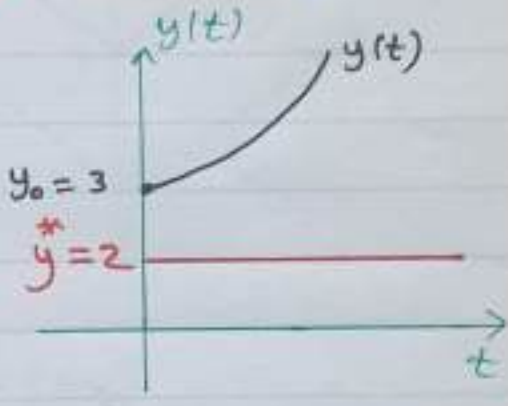
② substitute $y_0 = 3 \Rightarrow y' = 2 > 0 \Rightarrow y(t) \uparrow$

$y_0 = 1 \Rightarrow y' = -2 < 0 \Rightarrow y(t) \downarrow$

③ $\lim\limits_{t \to \infty} y(t) = \infty$



- The DF in Exp' shows the Integral Curves (IC's)
- These curves are all possible solutions for the DE $\Rightarrow$ They depend on the choice of y_0
- In part 3 when $y_0 = 3$ the DF becomes



$\rightarrow$ clearly $\lim\limits_{t\to\infty} y(t) = \infty$

$\rightarrow$ Here the DF contains only one Integral Curve which is the solution $y(t)$

- Back to Exp' $\Rightarrow$ we can see that the behavior of solution on y_0 as follow:

$$\lim_{t\to\infty} y(t) = \begin{cases} \infty & \text{if } y_0 > 2 \\ 2 & \text{if } y_0 = 2 \\ -\infty & \text{if } y_0 < 2 \end{cases}$$

- If we arrange the DE in Exp' in the form

$$y' = ay - b$$

$$y' = 2y - 4$$

then we can see that $a = 2$, $b = 4$
$+$

- Next example will be when $a < 0$ to see how the solution behave for different values of the initial condition y_0

$\underline{Exp}^2$ Consider the DE : $y' + 3y = 12$

① Draw the DF

② Find $\lim_{t\to\infty} y(t)$

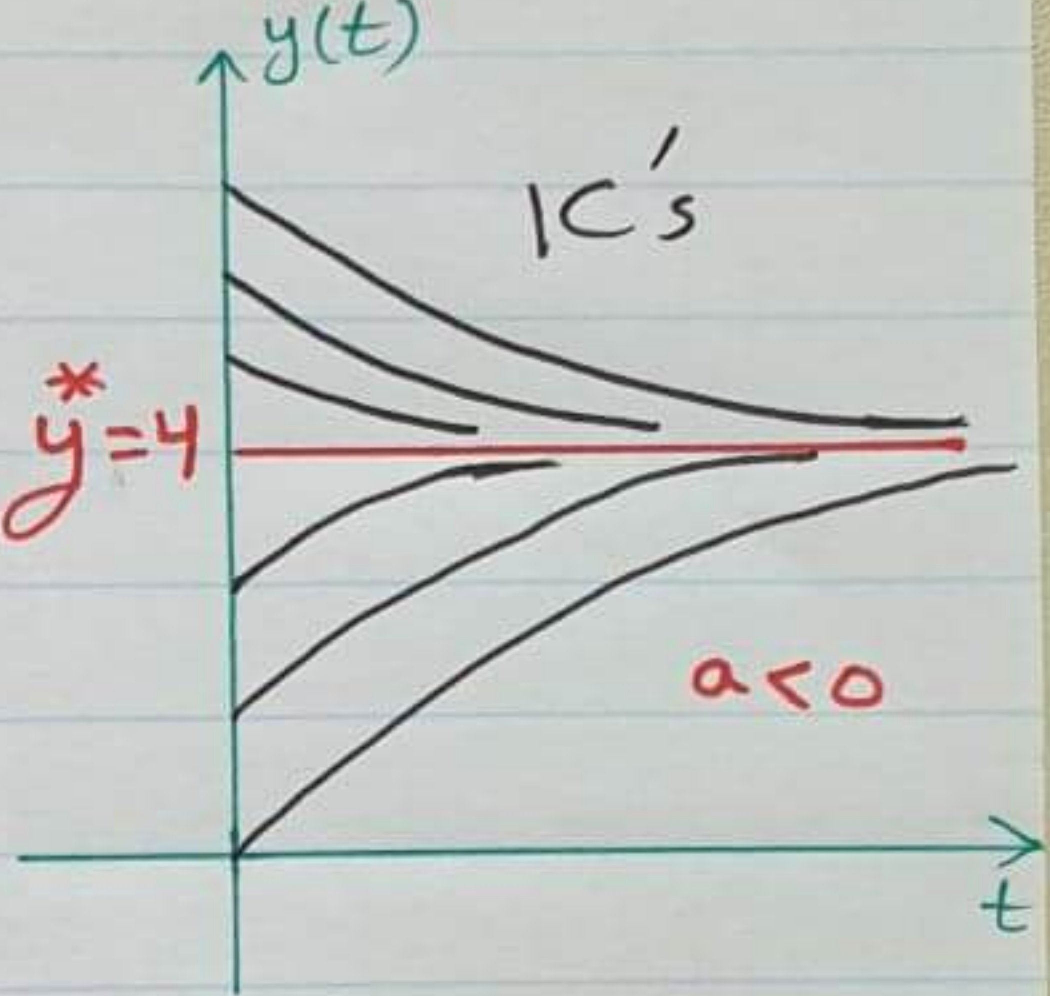
① First we find the Eq. Sol.

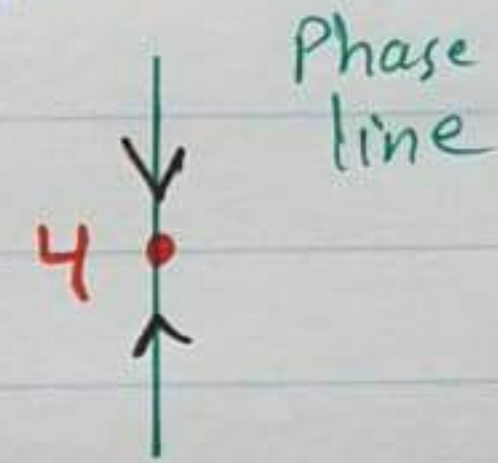
$\Rightarrow y' = 0 \Rightarrow 3y = 12$

$\Rightarrow y^* = 4$

substitue $y_0 = 5 \Rightarrow y' = -3 \Rightarrow y(t)\downarrow$

$y_0 = 0 \Rightarrow y' = 12 > 0 \Rightarrow y(t)\uparrow$





DF

② $\lim_{t\to\infty} y(t) = 4$

• Note that in Exp^2 can be arranged as

$y' = -3y + 12$ and comparing with

$y' = ay - b$ we see $a = -3$, $b = -12$

• In this Exp^2 the behavior of solution :

$$\lim_{t\to\infty} y(t) = \begin{cases} 4 & \text{if } y_0 > 4 \\ 4 & \text{if } y_0 = 4 \\ 4 & \text{if } y_0 < 4 \end{cases}$$

$$= 4 \quad \forall\ y_0$$

This is because $a < 0$.

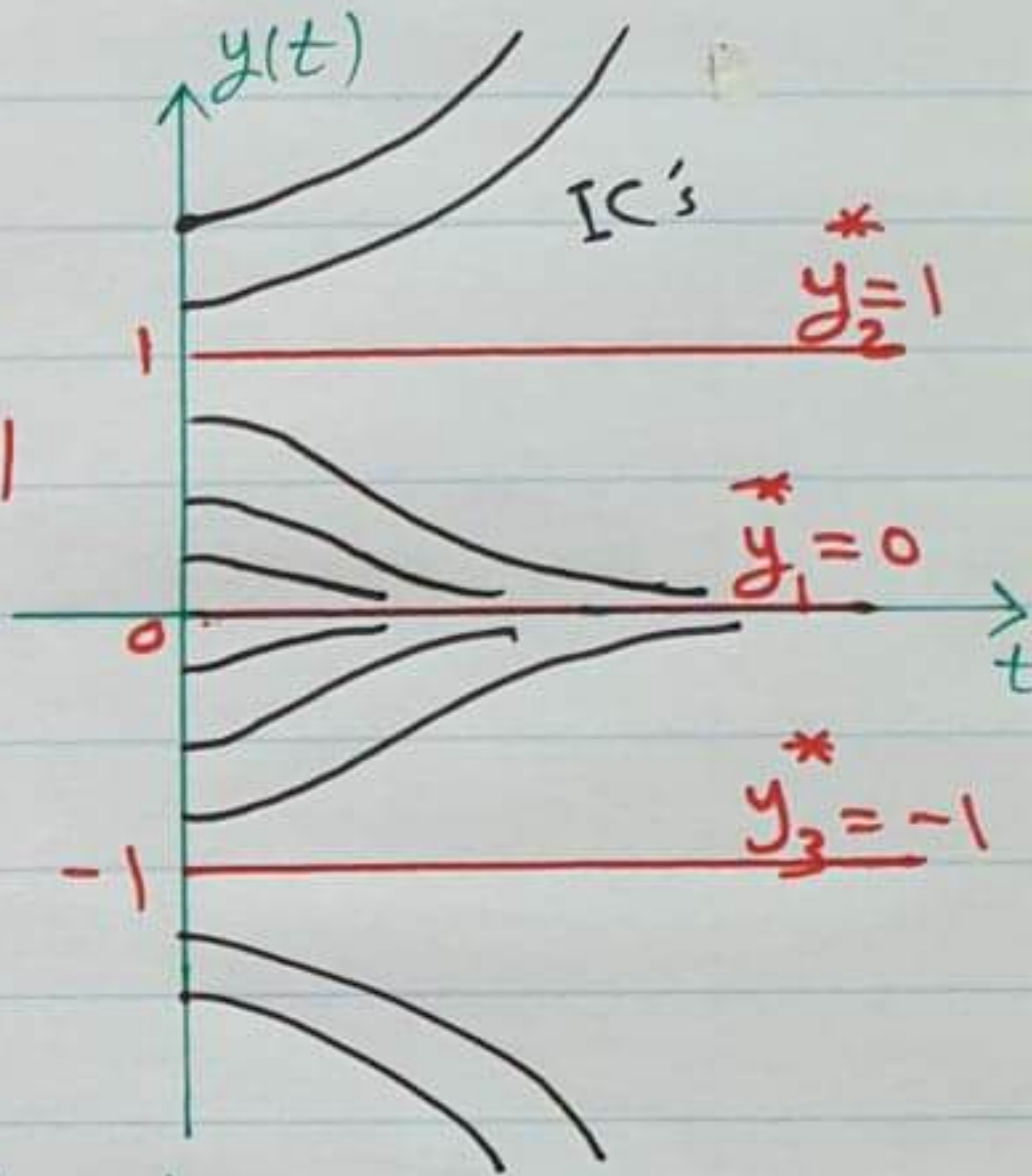
- Exp^1 and Exp^2 are examples of linear DE
- However, we can draw DF for some nonlinear DE's

$\underline{Exp^3}$ Consider the DE : $y' = y(y^2 - 1)$

① Find Eq. Sol.

② Draw DF

③ Find $\lim\limits_{t \to \infty} y(t)$ if $y(0) = 0.8$



① $y' = 0 \Rightarrow y(y-1)(y+1) = 0$

$y_1^* = 0,\ y_2^* = 1,\ y_3^* = -1$

② if $y_0 = 2 \Rightarrow y' = 6 > 0 \Rightarrow y(t) \uparrow$

if $y_0 = \frac{1}{2} \Rightarrow y' < 0 \Rightarrow y(t) \downarrow$

if $y_0 = -\frac{1}{2} \Rightarrow y' > 0 \Rightarrow y(t) \uparrow$

if $y_0 = -2 \Rightarrow y' < 0 \Rightarrow y(t) \downarrow$

Phase line

$\underline{DF}$

③ if $y_0 = 0.8 \Rightarrow \lim\limits_{t \to \infty} y(t) = 0$

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* In $Exp^3 \Rightarrow \forall\ y_0 \in (-1, 1) \Rightarrow$ the solution converges to zero

* The behaviour of solution:

$$\lim_{t \to \infty} y(t) = \begin{cases} \infty & \text{if } y_0 > 1 \\ 0 & \text{if } y_0 \in (-1,1) \\ -\infty & \text{if } y_0 < -1 \\ 1 & \text{if } y_0 = 1 \\ 0 & \text{if } y_0 = 0 \\ -1 & \text{if } y_0 = -1 \end{cases}$$

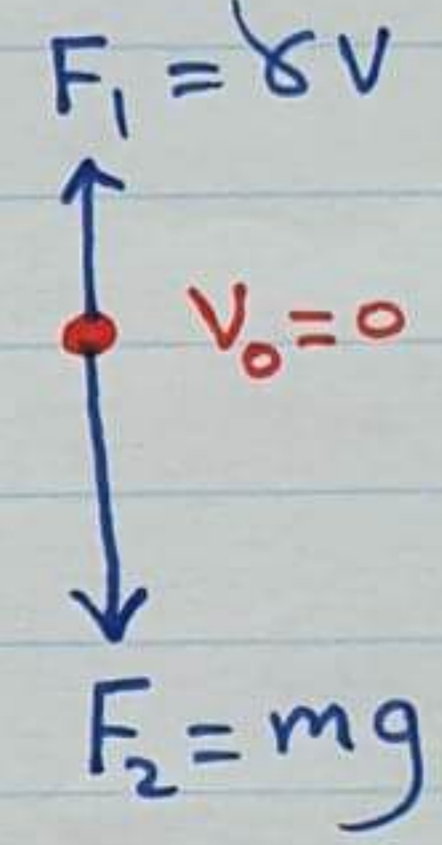
Modeling with DE's

- Sometimes DE's are called Mathematical Models (MM) as they can use to model (describe) a physical phenomena (waves, heats, velocity of objects, ...) or chemical phenomena (radioactivity, reactions, ...) or biological phenomena (population growth / Decay, ...) or psychological phenomena (process of making decisions, ...)

Exp (Free Fall)

An object falls from a rest

① write MM describing its velocity change over time



m : mass of object

$V(t)$: velocity of object at time t

γ : drg coefficient

$g \approx 9.8 \ m/sec^2$ acceleration due to gravity

Net force is $\Delta F = F_2 - F_1$

$$ma = mg - \gamma v$$

a: acceleration

$a = v'(t) = \frac{dv}{dt}$

$$m\frac{dv}{dt} = mg - \gamma v$$

$$\boxed{\frac{dv}{dt} = g - \frac{\gamma}{m} v} \rightarrow \text{MM}$$

② Take $m = 10$ Kg, $\gamma = 2$ Kg/sec, $g \approx 9.8 \ m/sec^2$

Find Eq. Sol.

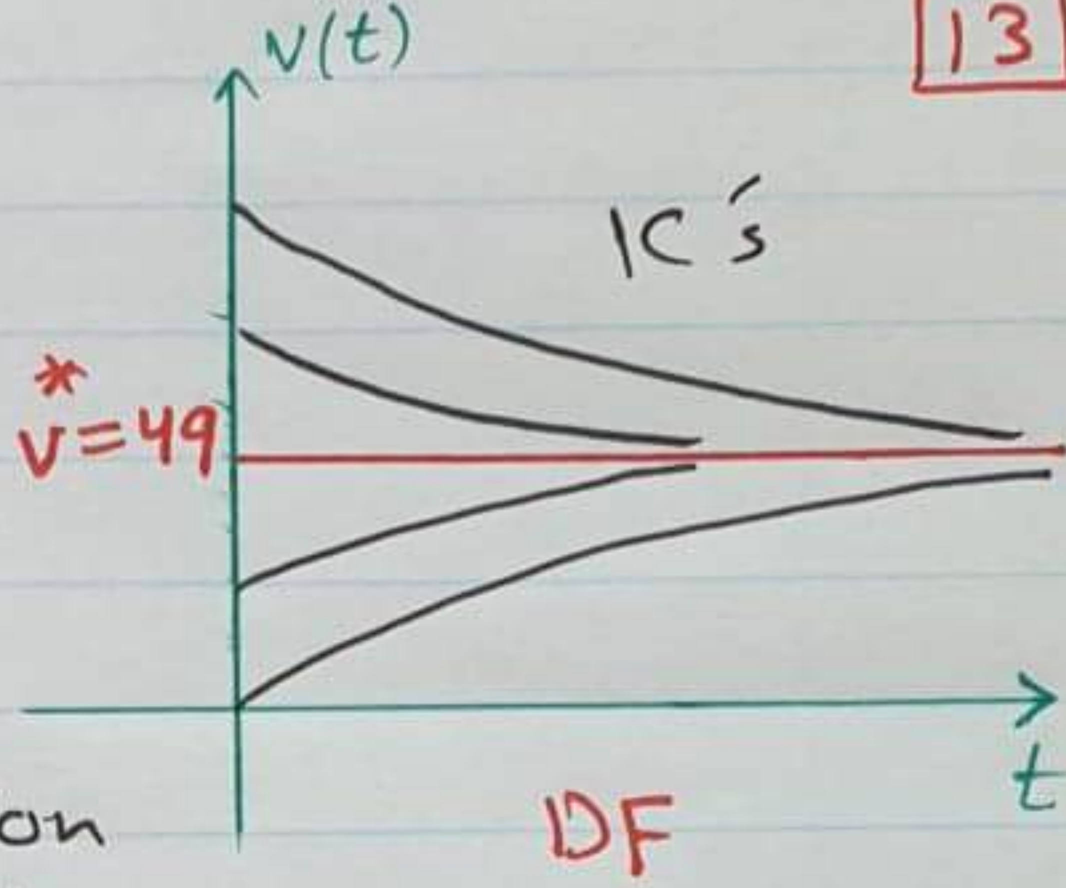
$$\frac{dv}{dt} = 9.8 - 0.2 v \quad \Rightarrow \frac{dv}{dt} = 0 \quad \Rightarrow v^* = \frac{9.8}{0.2} = 49$$

3) Draw the DF

$$\acute{v} = 9.8 - 0.2v$$

if $v_0 = 100$ then $\acute{v} < 0 \Rightarrow v(t) \downarrow$

if $v_0 = 0$ then $\acute{v} > 0 \Rightarrow v(t) \uparrow$



DF

4) study the behavior of solution "Find the limiting velocity"

$$\lim_{t \to \infty} v(t) = 49$$

$a = -0.2 < 0$

$v(t) \to 49$

Exp (Mice and Owls)

Assume a mice population $p(t)$ increases proportionally to its current size, where t in months.

1) write MM to describe the change in $p(t)$ over time

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = rP$$ where the constant r is called the growth rate ($r > 0$).

2) Assume the owls (predators) come and they eat 15 mice/day. Write MM to describe the change in $p(t)$ over time (consider $r = 0.5$)

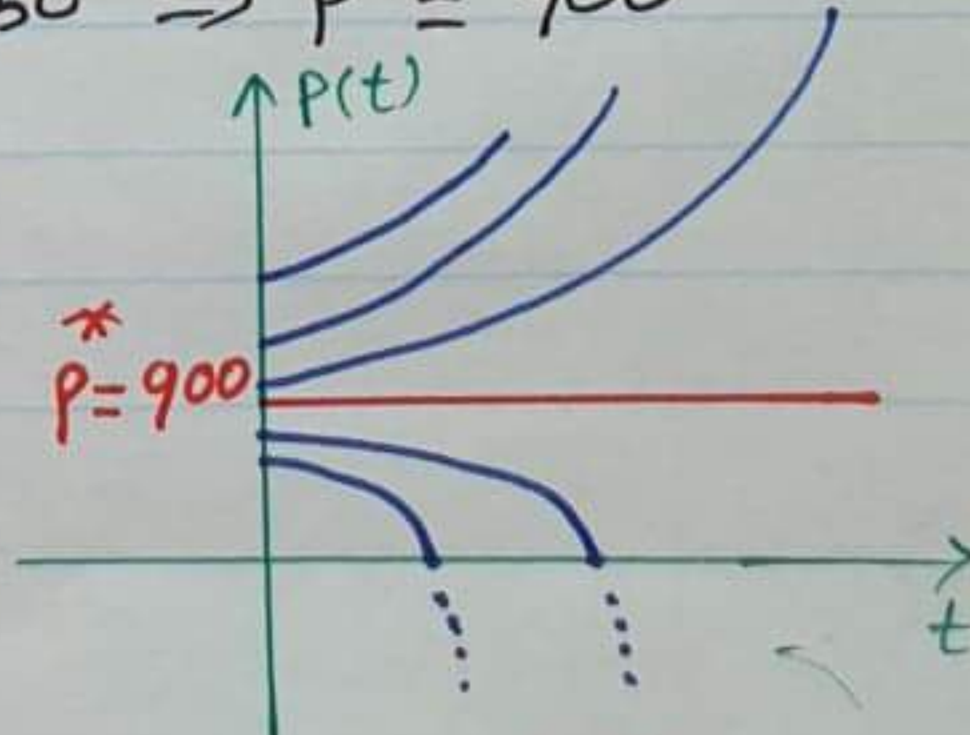
$$\frac{dP}{dt} = 0.5P - 450$$

15×30 since t in months

3) Draw the DF and study the behavior of solution

Eq. Sol. $\Rightarrow \frac{dP}{dt} = 0 \Rightarrow 0 = 0.5P - 450 \Rightarrow P^* = 900$

$$\lim_{t \to \infty} P(t) = \begin{cases} \infty & \text{if } P_0 > 900 \\ 900 & \text{if } P_0 = 900 \\ \boxed{0} & \text{if } P_0 < 900 \end{cases}$$



DF

Exp Assume $\phi(t)$ is a solution for the IVP:

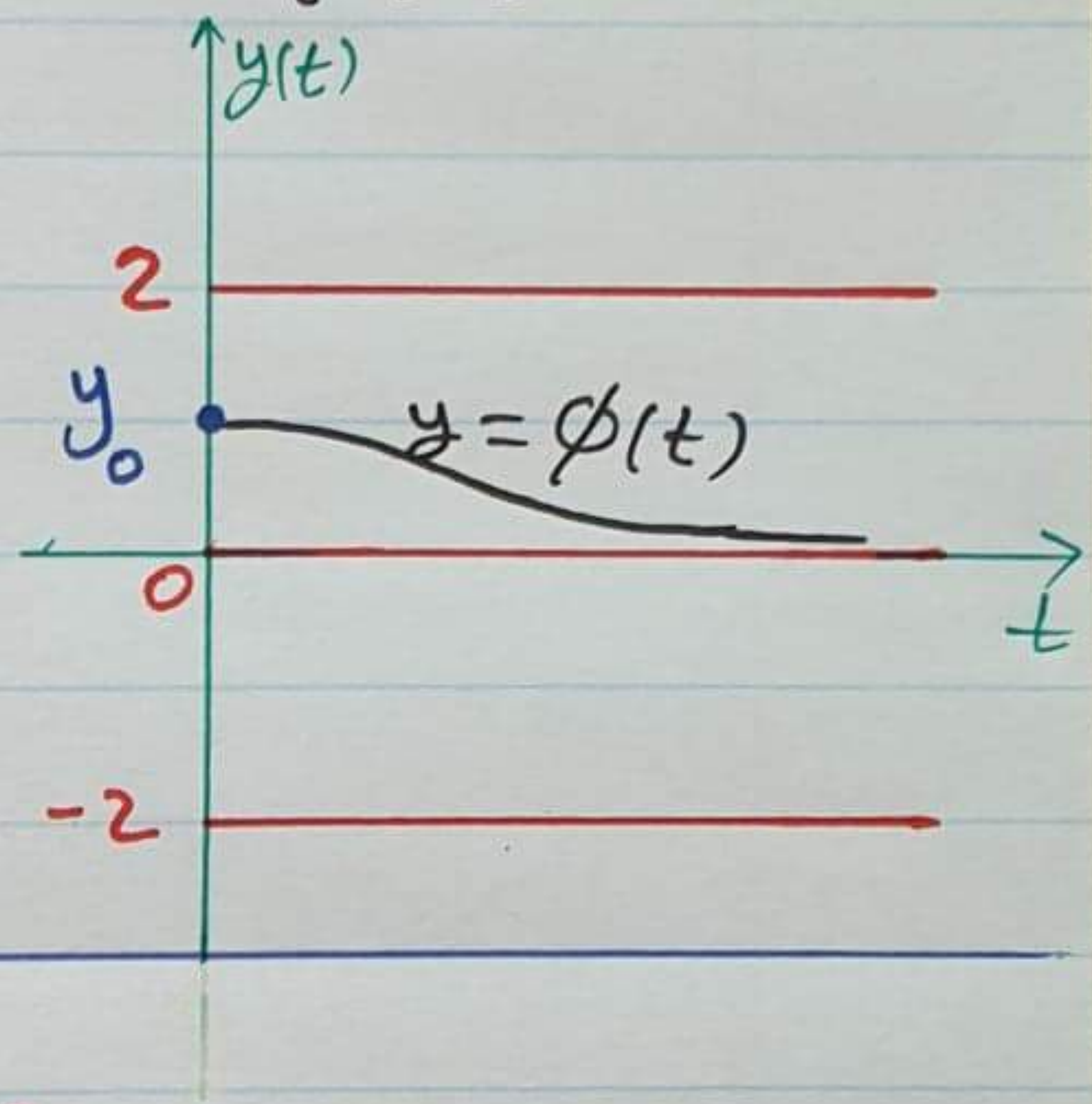
$$y' = y(y^2-4), \quad y(0)=1$$

Find $\lim_{t\to\infty} \phi(t)$

Eq. Sol. $\Rightarrow y' = 0 \Rightarrow y^* = 0, 2, -2$

If $y_0 = 1 \Rightarrow y' = -3 < 0 \Rightarrow y(t) \downarrow$

$$\lim_{t\to\infty} \phi(t) = 0$$



Three important questions for a given DE

1) Is there a solution?

2) If there is a sol., is it unique?

3) If there is a sol., how to find it?

- Now we will start solving DE's instead of drawing the DF
- The solution we get by solving a DE is called analatical solution

- Solution of some DE's

Exp How to solve 1^{st} order, linear IVP with constant coefficients and has the form:

$$\boxed{y' = ay - b \ , \ a \neq 0 \ , \ y(0) = y_0} \ \text{—} \ (A)$$

- We use the method of calulus :

$$\frac{dy}{dt} = ay - b$$

$$\frac{dy}{dt} = a\left(y - \frac{b}{a}\right) \Rightarrow \int \frac{dy}{y - \frac{b}{a}} = \int a \, dt$$

separable

$$\ln\left|y - \frac{b}{a}\right| = at + c$$

$$\left|y - \frac{b}{a}\right| = e^{at+c}$$

Note that $y^* = \frac{b}{a}$ is the Eq. Sol.

$$y - \frac{b}{a} = \pm e^{c} e^{at}$$

$$y(t) = \frac{b}{a} + D e^{at} \ , \quad D = \pm e^{c}$$

- To find the constant D, we use the IC $y(0) = y_0$

$$y(0) = \frac{b}{a} + D \Leftrightarrow y_0 = \frac{b}{a} + D \Leftrightarrow D = y_0 - \frac{b}{a}$$

- Hence, the solution of the IVP (A) is

$$\boxed{y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}} \longrightarrow A^*$$

- A^* is also called the general sol. because it contains all possible solutions, based on values of y_0.

Exp Solve the IVP:

① $\frac{dv}{dt} = 9.8 - 0.2V$, $V(\overset{t_0}{0}) = \overset{V_0}{0}$ " Free Fall"

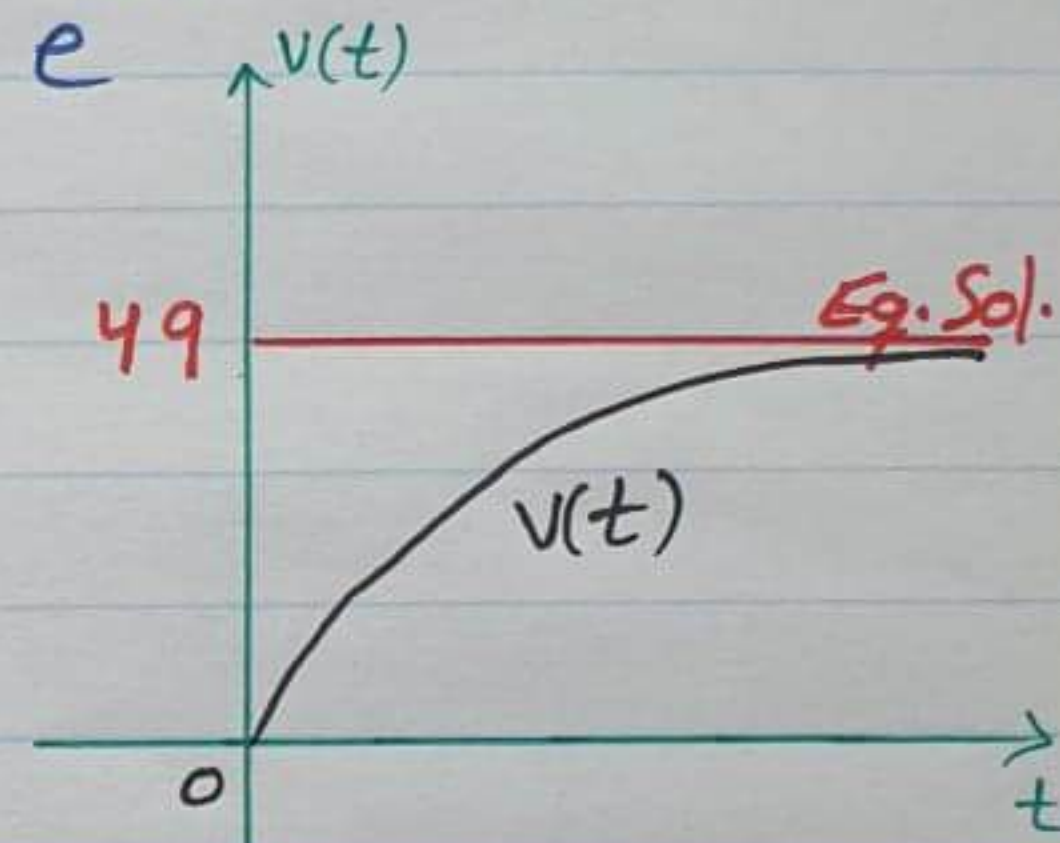
→ compare with Ⓐ : $y' = ay - b \Rightarrow a = -0.2$
$b = -9.8$

→ Apply A^* since the DE is 1^{st} order linear with constant coefficients

$$v(t) = \frac{b}{a} + \left(V_0 - \frac{b}{a}\right)e^{at}$$

$$= \frac{-9.8}{-0.2} + \left(0 - \frac{-9.8}{-0.2}\right)e^{-0.2t}$$

$$\boxed{V(t) = 49 - 49\,e^{-0.2t}}$$



Clearly $\lim_{t\to\infty} V(t) = 49$

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② $\frac{dP}{dt} = 0.5P - 450$, $P(0) = 850$ "Mice and Owls"

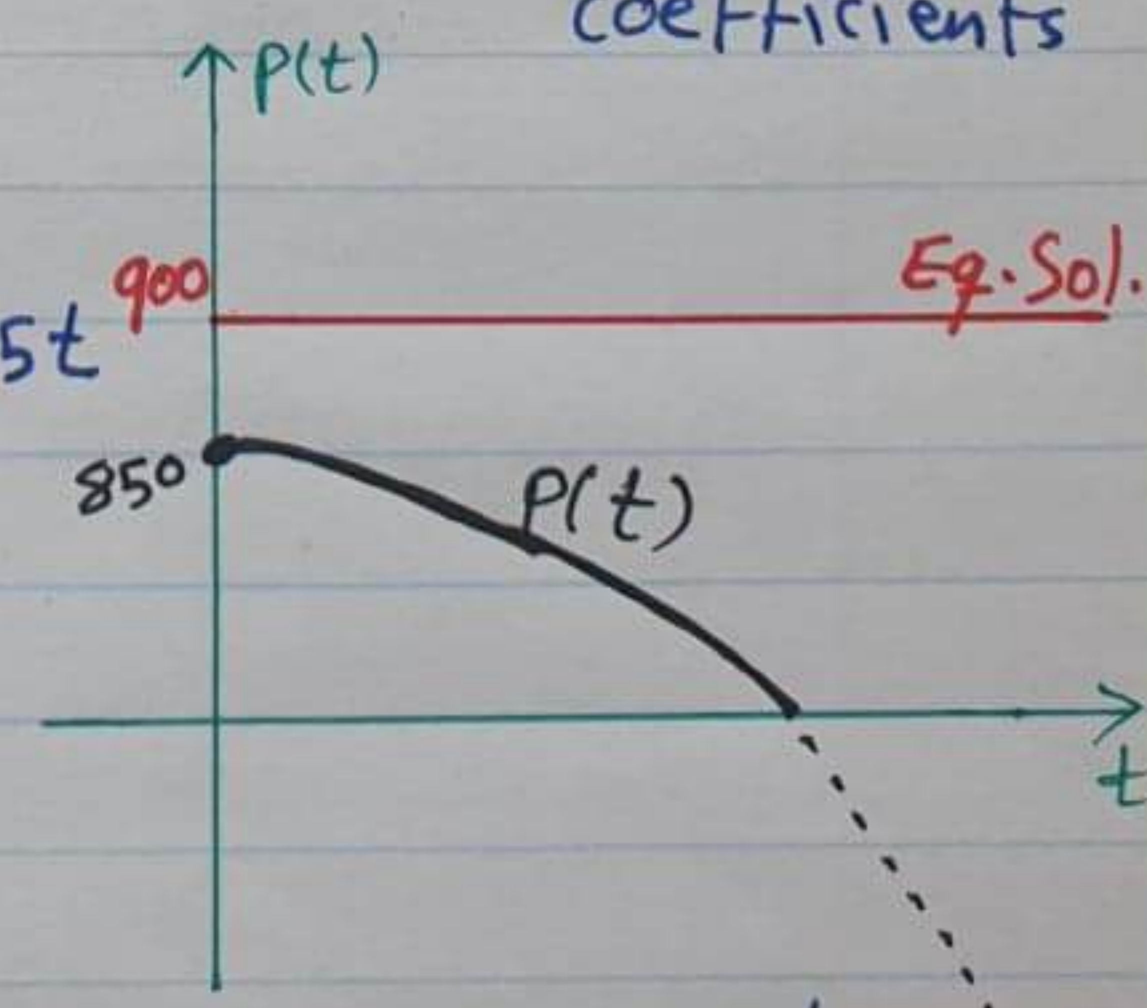
→ compare with Ⓐ : $y' = ay - b \Rightarrow a = 0.5$, $b = 450$

→ Apply A^* since the DE is 1^{st} order linear with constant coefficients

$$P(t) = \frac{b}{a} + \left(P_0 - \frac{b}{a}\right)e^{at}$$

$$= \frac{450}{0.5} + \left(850 - \frac{450}{0.5}\right)e^{0.5t}$$

$$\boxed{P(t) = 900 - 50\,e^{0.5t}}$$



$\lim_{t\to\infty} P(t) = 0$ since we talk about population

Exp Given the IVP: $y' - 2y = -4$, $y(0) = 3$. Find $y(\ln 2)$

- $y' = 2y - 4$ This DE satisfy Ⓐ with $a = 2, b = 4$

$\frac{b}{a} = 2$

- Apply A^* $\Rightarrow y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}$

$= 2 + (3 - 2)e^{2t}$

$\boxed{y(t) = 2 + e^{2t}}$ is the sol. of this IVP.

Now $y(\ln 2) = 2 + e^{2\ln 2} = 2 + e^{\ln 4} = 2 + 4 = 6$

Exp Assume a Bacteria population $p(t)$ increases proportionally to its current size. If the population doubles in two days, when it will tribble?

- $\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = aP$ a: growth rate
- This DE satisfy Ⓐ with $b = 0 \Rightarrow \frac{b}{a} = 0$
- Apply A^* to find the sol. $\Rightarrow$

$P(t) = \frac{b}{a} + \left(P_0 - \frac{b}{a}\right)e^{at} \Rightarrow \boxed{P(t) = P_0 e^{at}}$

- Since the population doubles in two days $\Rightarrow P(2) = 2P_0$

$P(2) = \cancel{P_0} e^{2a} = 2\cancel{P_0} \Leftrightarrow e^{2a} = 2 \Leftrightarrow 2a = \ln 2$

$\Leftrightarrow \boxed{a = \frac{\ln 2}{2}}$

- We need to find the time t^* s.t:

$P(t^*) = 3P_0$

$\cancel{P_0} e^{at^*} = 3\cancel{P_0}$

$e^{at^*} = 3 \Rightarrow at^* = \ln 3 \Rightarrow t^* = \frac{\ln 3}{a} = \frac{\ln 3}{\frac{\ln 2}{2}} = \frac{\ln 9}{\ln 2}$

Remark Assume y^* is an Eq. Sol. for the DE

$$y' = f(y) \quad \text{---} (1)$$

① If $f'(y^*) > 0$, then y^* is unstable Eq. Sol.

② If $f'(y^*) < 0$, then y^* is asymptotically stable Eq. Sol.

③ If $f'(y^*) = 0$, then y^* is semistable Eq. Sol.

Exp Find Eq. Sol's and classify them for the DE

① $y' = y(4-y^2) \Rightarrow y' = 0 \Rightarrow y(2-y)(2+y) = 0$

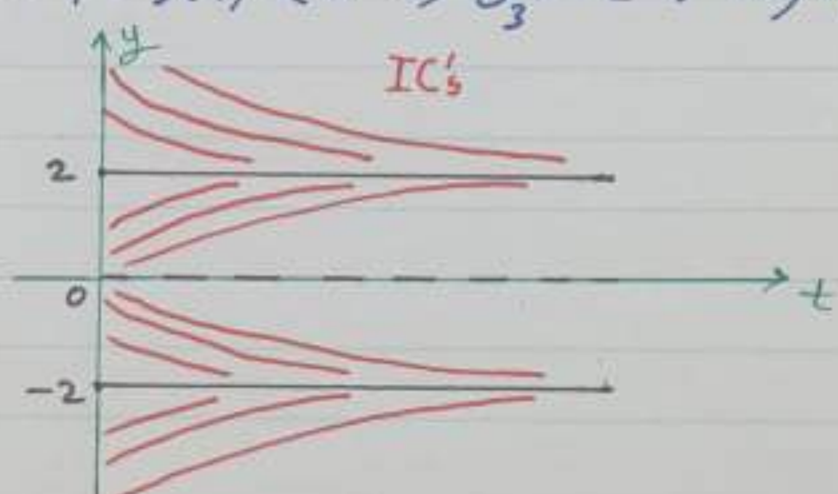
$\Rightarrow \underbrace{y_1^* = 0,\ y_2^* = 2,\ y_3^* = -2}_{\text{Eq. Sol}}$

Compare with (1) $\Rightarrow$

$y' = f(y) = y(4-y^2) = 4y - y^3$

$f'(y) = 4 - 3y^2$

- $f'(y_1^*) = f'(0) = 4 > 0 \Rightarrow y_1^* = 0$ is unstable Eq. Sol.
- $f'(y_2^*) = f'(2) = 4 - 3(4) < 0 \Rightarrow y_2^* = 2$ is asymptotically Eq. Sol.
- $f'(y_3^*) = f'(-2) = 4 - 3(4) < 0 \Rightarrow y_3^* = -2$ is asymptotically Eq. Sol.



DF

② $y' = y^2 \Rightarrow y' = 0 \Rightarrow y^* = 0$ Eq. Sol.

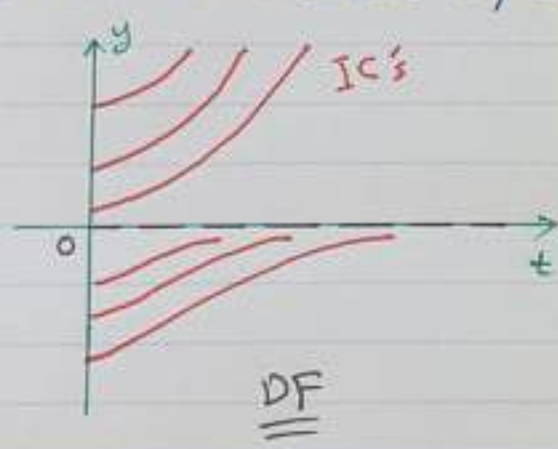
Compare with (1) $\Rightarrow$

$y' = f(y) = y^2 \Rightarrow f'(y) = 2y$

$f'(y^*) = f'(0) = 0 \Rightarrow y^* = 0$ is semistable Eq. Sol.

if $y_0 = 1 \Rightarrow y' > 0 \Rightarrow y(t) \uparrow$

if $y_0 = -1 \Rightarrow y' > 0 \Rightarrow y(t) \uparrow$



Notes ① If y^* is asymptotically stable Eq. Sol., then the IC's gets closer to y^* from both sides.

② If y^* is unstable Eq. Sol., then the IC's gets far away from y^* from both sides.

③ If y^* is semistable Eq. Sol., then the IC's gets closer to y^* from one side and becomes far away from y^* from the other side.

Radioactivity

Assume a radioactive material $Q(t)$ decays at rate proportional to the amount present at any particular time $\Rightarrow$

$$\boxed{\frac{dQ}{dt} = -r\,Q}$$, where $-r<0$ is the decay rate

This DE has the form of (A) "1^{st} order linear DE with constant coefficient" $\Rightarrow$ so we can use (A^*) to find the quantity $Q(t)$ available at time $t \Rightarrow$

$$\boxed{Q(t) = \frac{b}{a} + \left(Q_0 - \frac{b}{a}\right)e^{at}}$$ (A^*) where

$Q_0 = Q(0)$ is the initial amount of this material

$a = -r$ and $b = 0$ so $\frac{b}{a} = 0$

$$\boxed{Q(t) = Q_0\, e^{-rt}}$$ is the solution of this problem

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Half-life time τ : is the time takes the material to decay by the half of its initial

$$Q(\tau) = \frac{1}{2} Q_0$$

$$Q_0\, e^{-r\tau} = \frac{1}{2} Q_0$$

$$e^{-r\tau} = \frac{1}{2}$$

$$-r\tau = \ln\frac{1}{2}$$

$$-r\tau = -\ln 2$$

$$\boxed{\tau = \frac{\ln 2}{r}}$$

Note that

$-r<0 \Rightarrow r>0 \Rightarrow$
$\tau > 0$

17.4

Exp (Q14 - section 1.2)

Radium-226 has half-life of 1620 years.
Find the time period during which a given amount of this material is reduced by one-quarter.

$$\tau = \frac{\ln 2}{r} \Rightarrow r = \frac{\ln 2}{\tau} = \frac{\ln 2}{1620}$$

$Q(t) = Q_0 e^{-rt}$ is the amount available at time t

we need to find the time t^* s.t

$Q(t^*) = \frac{3}{4} Q_0$ since $\frac{1}{4}$ is reduced

$$Q_0 e^{-rt^*} = \frac{3}{4} Q_0$$

$$e^{-rt^*} = \frac{3}{4}$$

$$-rt^* = \ln \frac{3}{4}$$

$$t^* = \frac{\ln \frac{3}{4}}{-r} = \frac{\ln(0.75)}{-\frac{\ln 2}{1620}} = -1620 \frac{\ln(0.75)}{\ln 2}$$

$$\approx 672.4 \text{ years}$$