2.2: limits Theorem.

Thm 1: [squeeze Theorem]. sandwich Thm.

suppose that $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ are real sequences.

(i) If $x_n \to a$ and $y_n \to a$ (the same a) as $n \to \infty$, and if there is ($\exists$) an $N_0 \in \mathbb{N}$ s.t $\boxed{x_n \leq w_n \leq y_n \text{ for } n \geq N_0}$, then $w_n \to a$ as $n \to \infty$.

مهم كثير exp (ii) If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

$$\lim(x_n y_n) = \lim x_n \cdot \lim x_y \to \text{bdd}$$
$$= 0 \cdot \text{bdd}$$
$$= 0.$$

exp: e^{-n} and $\sin(n)$

$e^{-n} \to 0$ as $n \to \infty$

$\sin(n)$ bdd.

$e^{-n}(\sin(n)) = 0$.

proof:

(i) let $\varepsilon > 0$ be given. since x_n and y_n conv. to a

By def's, $\exists K_1, K_2 \in \mathbb{N}$ s.t

$n \geq K_1 \Rightarrow |x_n - a| < \varepsilon \;=\; a - \varepsilon < x_n < a + \varepsilon$

$n \geq K_2 \Rightarrow |y_n - a| < \varepsilon \;=\; a - \varepsilon < y_n < a + \varepsilon.$

let $K = \max\{N_0, K_1, K_2\}$

If $n \geq K$ we have $a - \varepsilon < x_n \underbrace{\leq w_n \leq}_{\text{By hypothisis}} y_n < a + \varepsilon \quad \forall n \geq K$

i.e $a - \varepsilon < w_n < a + \varepsilon$ for $n \geq K$

or $|w_n - a| < \varepsilon$, for $n \geq K$

We cnclude that $w_n \to a$ as $n \to \infty$

(ii) spse that $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bdd.
since $\{y_n\}$ is bdd then $\exists M > 0$ s.t $|y_n| \leq M$, $\forall n \in \mathbb{N}$.

let $\varepsilon > 0$. since $x_n \to 0$ as $n \to \infty$
$\exists$ a $K \in \mathbb{N}$ s.t $n \geq K \Rightarrow |x_n| < \frac{\varepsilon}{M}$.
Then $n \geq K \Rightarrow |x_n y_n - 0| = |x_n||y_n|$

$$< \frac{\varepsilon}{\cancel{M}} \cancel{M}$$
$$< \varepsilon .$$

$\therefore x_n y_n \to 0$ as $n \to \infty$ ▭

exp: Find $\lim_{n\to\infty} \frac{\cos(n^3 - n^2 + n - 13)}{2^n}$ By sandwich Theorem

since $|\cos x| \leq 1$, $\forall x \in \mathbb{R}$
Then
$$\left| \frac{\cos(n^3 - n^2 + n - 13)}{2^n} \right| \leq 2^{-n}$$

$$\overset{x_n}{-\frac{1}{2^n}} < \overset{w_n}{\frac{\cos(n^3 - n^2 + n - 13)}{2^n}} < \overset{y_n}{\frac{1}{2^n}}$$

since $\lim_{n\to\infty} \frac{-1}{2^n} = \lim_{n\to\infty} \frac{1}{2^n} = 0$ (def'n do it).

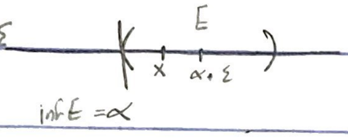
then $\lim_{n\to\infty} \frac{\cos(n^3 - n^2 + n - 13)}{2^n} = 0$

By squeeze theroem

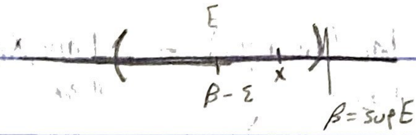
Thm 2: let $E \subseteq \mathbb{R}$ if E has a finite supremum (respectively, a finite infimum). Then there is a sequence $x_n \in E$ s.t $x_n \to \sup E$ (respectively, a sequence $y_n \in E$ s.t $y_n \to \inf E$) as $n \to \infty$.

مهمة كثير كثير

$\exists \alpha \in E : \alpha \leq x < \alpha + \varepsilon$



/ $\exists x \in E : \beta - \varepsilon < x \leq \beta$



exp: $E = (0,1) \to \sup E = 1$, $\inf E = 0$

$x_n = \frac{1}{n} \in E$ and $x_n \to 0 = \inf E$.

$y_n = \frac{n}{n+1} \in E$ and $y_n \to 1 = \sup E$.

proof: spse that $\sup E = \beta < +\infty$,

By the Approximation property for suprema ($\exists x \in E$ s.t $\beta - \varepsilon < x \leq \beta$, $\forall \varepsilon > 0$).

$\exists\, x_1 \in E : \beta - 1 < x_1 \leq \beta$

$\exists\, x_2 \in E : \beta - \frac{1}{2} < x_2 \leq \beta$

$\exists\, x_3 \in E : \beta - \frac{1}{3} < x_3 \leq \beta$

$\vdots$

$$\exists\, x_n \in E : \underbrace{\beta - \frac{1}{n}}_{\beta \text{ as } n\to\infty} < x_n \leq \beta \to \beta \text{ as } n \to \infty$$

Then by the squeeze thm (since $\lim_{n\to\infty}(\beta - \frac{1}{n}) = \lim_{n\to\infty} \beta = \beta$ Then

$\lim_{n\to\infty} x_n = \beta = \sup E$.

similarly, $\exists$ a seq. $y_n \in E$ s.t $\inf E \leq y_n < \underbrace{\inf E + \overset{\varepsilon}{\frac{1}{n}}}_{\to \inf E \text{ as } n\to\infty}$

Then, By the squeeze thm, $\lim_{n\to\infty} y_n = \inf E$

Thm 3: suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences and that $\alpha \in \mathbb{R}$.
If $\{x_n\}$ and $\{y_n\}$ are convergent, then

(i) $\lim\limits_{n\to\infty} (x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$.

(ii) $\lim\limits_{n\to\infty} (\alpha x_n) = \alpha \lim\limits_{n\to\infty} x_n$

(iii) $\lim\limits_{n\to\infty} (x_n y_n) = (\lim\limits_{n\to\infty} x_n)(\lim\limits_{n\to\infty} y_n)$.

→ If, in addition, $y_n \neq 0$ and $\lim\limits_{n\to\infty} y_n \neq 0$, then

(iv) $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$.

(In particular, all these limits exist).

Proof: spse that $x_n \to x$ and $y_n \to y$ as $n \to \infty$

(i) let $\varepsilon > 0$ be given, since $x_n \to x$, $\exists K \in \mathbb{N}$ s.t $|x_n - x| < \frac{\varepsilon}{2}$ $\forall n \geq K$
since $y_n \to y$, $|y_n - y| < \frac{\varepsilon}{2}$, $\forall n \geq K$.
Thus, $n \geq K$ implies

$$\begin{aligned} |(x_n + y_n) - (x+y)| &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y|. \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

we conclude that $x_n + y_n \to x + y$ as $n \to \infty$

(ii) Given $\lim\limits_{n\to\infty} x_n = x$ conv., $\alpha \in \mathbb{R}$, we need to show $\alpha x_n \to \alpha x$ as $n\to\infty$.

It satifficies to show $\alpha x_n - \alpha x \to 0$ as $n \to \infty$

since $x_n - x \to 0$ as $n\to\infty$ $\{\alpha\}$ is bdd,

By squeeze thm, $\alpha(x_n - x) \to 0$ as $n\to\infty$

أثبت باستخدام (squeeze thm) و بدي أستخدم التعريف بكون عندي 2 casses.

(iii) $\lim\limits_{n\to\infty} (x_n y_n) = xy$, It safficies to show $x_n y_n - xy \to 0$ as $n\to\infty$.

Now, $x_n y_n - xy = \underbrace{x_n}_{\text{bdd since it is conv.}} \underbrace{(y_n - y)}_{\overset{0}{\text{given}}} + \underbrace{(x_n - x)}_{\overset{0}{\text{given}}} \, y \to$ bdd (constant).

$\longrightarrow$ $0 + 0$ as $n \to \infty$ By squeeze thm

$\therefore x_n y_n \to xy$ as $n \to \infty$

(iv) $\dfrac{x_n}{y_n} - \dfrac{x}{y} = \dfrac{x_n y - x y_n}{y y_n} = \dfrac{x_n y - xy + xy - x y_n}{y y_n}$

Thus, $\left|\dfrac{x_n}{y_n} - \dfrac{x}{y}\right| \leq \dfrac{|x_n - x|}{y_n} + \dfrac{|x|}{|y y_n|} |y_n - y|$.

since $y \neq 0$, $|y_n| \geq \dfrac{|y|}{2}$ for large n. Thus

$\left|\dfrac{x_n}{y_n} - \dfrac{x}{y}\right| \leq \dfrac{2}{|y|} |x_n - x| + \dfrac{2\,|x|}{|y|^2} |y_n - y| \longrightarrow \infty$

as $n\to\infty$ (By Thm 3 i and ii) and squeeze Thm $\dfrac{x_n}{y_n} \to \dfrac{x}{y}$ as $n \to \infty$

Def 1: Let $\{x_n\}$ be a sequence of real numbers

(i) $\{x_n\}$ is said to diverge to $+\infty$ (notation $x_n \to +\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = +\infty$) iff for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ s.t $n \geq N$ implies $x_n > M$.

$x_n = n^2$

$\lim_{n\to\infty} (n^2) = \infty$

$M = 100$

$n \geq 11$

$x_n = 121 > 100$

$M = 20$

$\exists N : n \geq N \Rightarrow x_n > 20$

(ii) $\{x_n\}$ is said to diverge to $-\infty$ (notation $x_n \to -\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = -\infty$) iff for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ s.t $n \geq N$ implies $x_n < M$

exp: prove that $\lim_{n\to\infty} (n^2 - n) = \infty$.

proof: Let $M \in \mathbb{R}$. use Archimedean principle, $\exists$ an $N \in \mathbb{N}$ s.t $\boxed{N > \max\{M, 2\}}$

Then $n \geq N \Rightarrow x_n = n^2 - n = n(n-1) > N(N-1) > M(1)$

$= M(2-1)$

$= M$

هلأ جبت

$N(N-1) > M$

if $\boxed{N > M}$ and

$N - 1 > 1$

$\boxed{N \geq 2}$

b. prove that $\lim_{n\to\infty} (n - 3n^2) = -\infty$ By def.

Proof: let $M \in \mathbb{R}$. use Archimedean principle, $\exists$ an $N \in \mathbb{N}$ s.t $N > \frac{-M}{2}$

Then $n \geq N \Rightarrow x_n = n - 3n^2$

$= n(1 - 3n)$

$\leq -2n$

$\leq -2N < M$

$\frac{-2N}{-2}$ $\frac{M}{-2}$

$n \geq 1$

$-3n \leq -3$

$1 - 3n \leq -2$

$\boxed{N > \frac{M}{-2}}$

exp: prove that $\lim_{n\to\infty}\left(\overset{\frac{n^2+2}{2n}}{\frac{n}{2}+\frac{1}{n}}\right)=\infty$

proof: let $M\in\mathbb{R}$, By Archimedean principle, $\exists$ an $N\in\mathbb{N}$ s.t $N>2M$.

Then $n\geq N \Rightarrow x_n=\frac{n}{2}+\underbrace{\frac{1}{n}}_{\geq 0}\geq\frac{n}{2}\geq\frac{N}{2}>\frac{2M}{2}=M$

$\boxed{=}$

Thm 4: suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences s.t $x_n\to+\infty$ (respectively, $x_n\to-\infty$) as $n\to\infty$

(i) If y_n is $\overset{y_n\geq M}{\underline{\text{bounded below}}}$ (serpectively, y_n is $\overset{y_n\leq M}{\underline{\text{bounded above}}}$), then

$\lim_{n\to\infty}(x_n+y_n)=+\infty$ (respectively, $\lim_{n\to\infty}(x_n+y_n)=-\infty$).

(ii) If $\alpha>0$, then

$\lim_{n\to\infty}(\alpha x_n)=+\infty$ (respectively, $\lim_{n\to\infty}(\alpha x_n)=-\infty$).

(iii) If $y_n>M_0$ for some $M_0>0$ and all $n\in\mathbb{N}$, then

$\lim_{n\to\infty}(x_ny_n)=+\infty$ (respectively, $\lim_{n\to\infty}(x_ny_n)=-\infty$).

(iv) If $\{y_n\}$ is bounded and $x_n\neq 0$ then

$\lim_{n\to\infty}\frac{y_n}{x_n}=0$.

Proof Thm 4: supose $x_n \to +\infty$ as $n \to \infty$

(i) By hypothesis, $y_n \geq M_0$ for some $M_0 \in \mathbb{R}$. (the case $x_n \to -\infty$ similarly).

let $M \in \mathbb{R}$ and set $\underline{M_1 = M - M_0}$.

since $x_n \to +\infty$, $\exists N \in \mathbb{N}$ s.t $n \geq N \Rightarrow x_n > M_1$

Then $n \geq N \Rightarrow x_n + y_n > \underbrace{M_1}_{M-M_0} + M_0 = \underline{M}$.

(ii) let $M \in \mathbb{R}$ and set $M_1 = \frac{M}{\alpha}$.

since $x_n \to +\infty$, $\exists N \in \mathbb{N}$ s.t $n \geq N \to x_n > M_1$

since $\alpha > 0$ we conclude that $\underline{\alpha x_n > \alpha M_1 = M}$ $\forall n \geq N$

(iii) let $M \in \mathbb{R}$ and set $M_1 = \frac{M}{M_0}$

since $x_n \to +\infty$, $\exists N \in \mathbb{N}$ s.t $n \geq N \Rightarrow x_n > M_1$

Then $n \geq N \Rightarrow x_n y_n > M_1 y_n > M_1 M_0 = M$.

(iv) let $\varepsilon > 0$. since $\{y_n\}$ is bdd, then $\exists M_0 > 0$ s.t $|y_n| \leq M_0$.

since $x_n \to +\infty$ as $n \to +\infty$

Then $\exists N \in \mathbb{N}$ s.t $n \geq N \Rightarrow x_n > M_1$

Choose M_1 so large that $\frac{M_0}{M_1} < \varepsilon$.

Then $n \geq N \Rightarrow \left|\frac{y_n}{x_n}\right| = \frac{|y_n|}{x_n} < \frac{M_0}{M} < \varepsilon$

RMK:

1. $x+\infty=\infty$, $x-\infty=-\infty$, $\forall x\in\mathbb{R}$.

2. $x.\infty=\infty$, $x.-\infty=-\infty$, $x>0$.

3. $x.\infty=-\infty$, $x.-\infty=\infty$, $x<0$.

4. $\infty+\infty=\infty$, $-\infty-\infty=-\infty$.

5. $\infty.\infty=(-\infty).(-\infty)=\infty$ And
$(-\infty).\infty=\infty.(-\infty)=-\infty$

Corollary: let $\{x_n\}$, $\{y_n\}$ be real sequences and α, x, y be extended real numbers
If $x_n\to x$ and $y_n\to y$ as $n\to\infty$, then

$$\lim_{n\to\infty}(x_n+y_n)=x+y .$$

provided that the right side is not of the form $\infty-\infty$, and

$$\lim_{n\to\infty}(\alpha x_n)=\alpha x \quad , \quad \lim_{n\to\infty}(x_n y_n)=xy .$$

provided that none of these products is of the form $0.\mp\infty$.

exp: $x_n=n^2\to\infty$

$y_n=n\to\infty$

$x_n+y_n=\infty+\infty$

$=\infty$

$$\lim_{n\to\infty}\left(n.\frac{1}{n}\right)\neq\lim_{n\to\infty}n\cdot\lim_{n\to\infty}\frac{1}{n} .$$

But $\lim_{n\to\infty}\left(n.\frac{1}{n}\right)=\lim_{n\to\infty}(1)=1$.

exp: $\lim_{n\to\infty} (n^2-n) \neq \lim_{n\to\infty} n^2 - \lim_{n\to\infty} n$

But $\lim_{n\to\infty} (n^2-n) = \lim_{n\to\infty} n^2(1-\frac{1}{n})$

$= \lim_{n\to\infty} n^2 \cdot \lim_{n\to\infty} (1-\frac{1}{n})$

$= \infty \cdot 1$

$= \infty$.

Thm 5 [Comparison Theorem].

suppose that $\{x_n\}$ ($x_n \to x$) and $\{y_n\}$ ($y_n \to y$) are convergent sequences. If there is an $N_0 \in \mathbb{N}$

s.t $\boxed{x_n \leq y_n \quad \text{for} \quad n \geq N_0 \quad \text{Then} \quad x = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n = y}$

In particular, if $x_n \in [a,b]$ converge to some point c, then c must belong to $[a,b]$.

$a \leq x_n \leq b \Rightarrow \lim_{n\to\infty} a \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} b$

$a \leq c \leq b$

proof:

supose that the first statement is false, i.e, that $x_n \leq y_n$ for $n \geq N_0$

holds but $x := \lim_{n\to\infty} x_n > \lim_{n\to\infty} y_n := y$

set $\varepsilon = \frac{x-y}{2}$. choose $N > N_0$ s.t $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ $\forall n \geq N_1$.

Then for such an $n \geq N$,

$$x_n > x - \varepsilon = x - \left(\frac{x-y}{2}\right) = y + \left(\frac{x-y}{2}\right) = y + \varepsilon > y_n$$

$\Rightarrow x_n > y_n$, which contradicts ($x_n \leq y_n$ for $n \geq N_0$).

This prove the first statment

cont pf: To prove second statment, we conclude

$a \leq x_n \leq b$ then by first statement

$$\lim_{n\to\infty} a \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} b$$

This implies $a \leq c \leq b$ ∎

شلنا المساواة

RMK: $x_n \underline{<} y_n$, $n \geq N_0$ Does Not imply that $\lim\limits_{n\to\infty} x_n < \lim\limits_{n\to\infty} y_n$.

counter exp: $\frac{1}{n^2} < \frac{1}{n}$ But $\lim\limits_{n\to\infty} \frac{1}{n^2} \not< \lim\limits_{n\to\infty} \frac{1}{n}$.