4.2: Differentiability Theorems.

Thm 4: let $I \subseteq \mathbb{R}$ be an interval, let $a \in I$, $\alpha \in \mathbb{R}$ and let $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ be functions that diffble at a, then $f+g$, αf, $f.g$ and (when $g(a) \neq 0$) $\frac{f}{g}$ are all diffble at a. In fact:

i. $(f+g)'(a) = f'(a) + g'(a)$.

ii. $(\alpha f)'(a) = \alpha f(a)$.

iii. $(f.g)'(a) = g(a) f'(a) + f(a) g'(a)$.

IV. $\left(\frac{f}{g}\right)'(a) = \dfrac{g(a)\, f'(a) - f(a)\, g'(a)}{g^2(a)}$,

→ proof on Notes.

pf IV:

let $q = \frac{f}{g}$, since g is diffble at a, it is continuous at a. (Thm3), since $g(a) \neq 0$ $\exists$ an interval $J \subseteq I$ with $a \in J$ s.t $g(x) \neq 0$, $\forall x \in J$. For $x \in J$, $x \neq a$, we have

$$\left(\frac{f}{g}\right)'(a) = q'(a) = \lim_{x \to a} \frac{q(x) - q(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x - a}$$

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$$= \lim_{x \to a} \frac{f(x)\, g(a) - f(a)\, g(x)}{(x-a)\, g(x)\, g(a)}$$

$$= \lim_{x \to a} \frac{f(x) g(a) - f(a) g(a) + f(a) g(a) - f(a) g(x)}{(x-a)\, g(x)\, g(a)}$$

we used the continuity of g at a and the differentiability of f and g at a.

$$= \lim_{x \to a} \frac{1}{g(x)\, g(a)} \left[\frac{f(x) - f(a)}{x - a} g(a) - f(a) \cdot \frac{g(x) - g(a)}{x - a} \right]$$

$$= \frac{1}{g^2(a)} \left[f'(a)\, g(a) - f(a)\, g'(a) \right]$$

proof by induction.

Corollary : If $f_1, f_2, \dots f_n$ are functions on an interval I to $\mathbb{R}$ that are diffble at $a \in I$, then :

i. The function $f_1 + f_2 + \dots + f_n$ is diffble at a and ← [illegible]

$$(f_1 + f_2 + f_3 + \dots f_n)'(a) = f_1'(a) + f_2'(a) + \dots + f_n'(a).$$

ii. The function $f_1 f_2 \dots f_n$ is diffble at a, and $(f_1 f_2 \dots f_n)'(a) =$

$$f_1'(a) f_2(a) \dots f_n(a) + f_1(a) f_2'(a) \dots f_n(a) + \dots + f_1(a) f_2(a) \dots f_n'(a).$$

$$(f_1 f_2 \dots f_{n+1})' = (f_1 \dots f_n \, f_{n+1})' = \underbrace{(f_1 \dots f_n)'}_{\text{induction}} f_{n+1} + (f_1 \dots f_n) f_{n+1}'$$

Thm 5 : chain Rule .

let f and g be real functions If f is diffble at a and g is diffble at $f(a)$, then $g \circ f$ is diffble at a with

$$(g \circ f)'(a) = g'(f(a)) \, f'(a) .$$

$$\left[g(f(x)) \right]' \Big|_{x=a} = g'(f(x)) \Big|_{x=a} f'(x) \Big|_{x=a}$$

$$= g'(f(a)) \, f'(a)$$ $\circ \to 5, 8, 9$

Proof →

proof :

By Thm 1, $\exists$ an open intervals I and J and $F: I \to \mathbb{R}$ cont. at a and $G: J \to \mathbb{R}$ cont. at $f(a)$, such that

$F(a) = \bar{f}(a)$, $G(f(a)) = \bar{g}(f(a))$,

$f(x) = F(x)(x-a) + f(a)$, $\forall x \in I$ ---- i

$g(y) = G(y)(y - f(a)) + g(f(a))$, $\forall y \in J$. ---- ii

since f is cont. at a we may assume that $f(x) \in J$, $\forall x \in I$

Fix $x \in I$, Apply ii to $y = f(x)$ and i to x to write

$$(g \circ f)(x) = g(f(x))$$

By ii

$$= G(f(x)) \underbrace{(f(x) - f(a))}_{} \overset{\text{By i}}{+} g(f(a))$$

$$= G(f(x)) F(x) (x-a) + (g \circ f)(a).$$

Set $H(x) = G(f(x)) F(x)$ for $x \in I$

since F is cont. at a and G is cont. at $f(a)$, then H is cont. at a.

Moreover, $H(a) = G(f(a)) F(a)$

$H(a) = \bar{g}(f(a)) \bar{f}(a)$

It follows from Thm 1, $(g \circ f)'(a) = H(a)$,

i.e, $(g \circ f)'(a) = \bar{g}(f(a)) \bar{f}(a)$