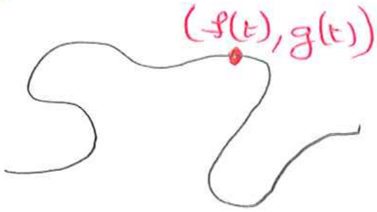
Chapter 11. Parametric Equations and Polar Coordinates.

11.1 Parametrization of Plane Curves.

We may describe the movement of a particle in the xy plane at position t by

$$(x(t), y(t)) = (f(t), g(t))$$



Notice that the position of the particle at time t is not a function.

Def : If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an Interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve.

- The equations are parametric equations of the curve.

- The variable t is called the parameter of the curve.

- I is called the parameter Interval.
- If $I = [a, b]$ is a closed interval, the point $(f(a), g(a))$ is the initial point of the curve and the point $(f(b), g(b))$ is the terminal point.
- We say that we parametrized the curve, if we find ① $x = f(t)$, ② $y = g(t)$ and ③ I that is, we find a parametrization of the curve.

Example: Given the parametric equations

$$x = t^2 \quad , \quad y = t+1 \quad , \quad -\infty < t < \infty .$$

1) Find the Cartesian (algebraic) equation by eliminating the paremeter t.

2) Identify the particle's path by sketching the equations.

3). Find the direction of motion.

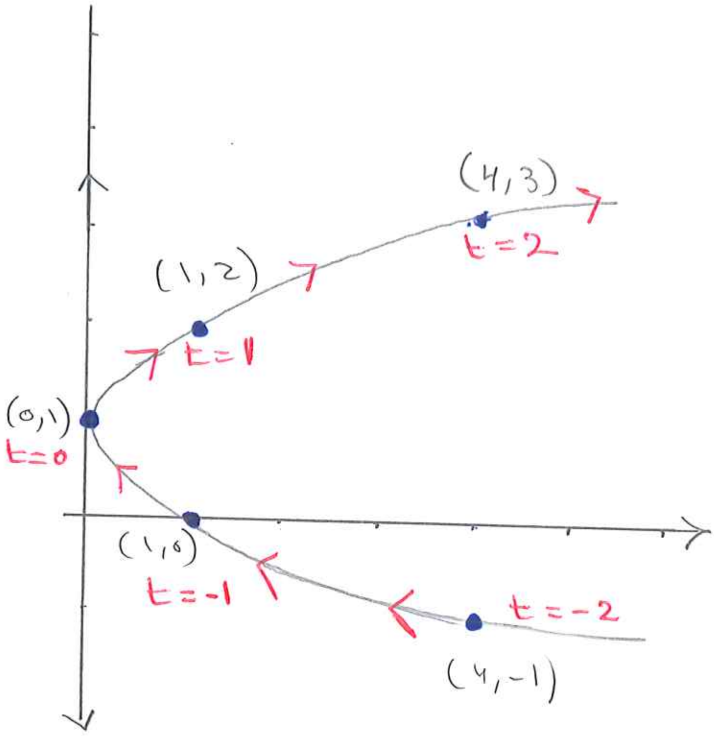
Sol: 1) Cartesian equation :

$$x = t^2 = (y-1)^2 \Rightarrow x = (y-1)^2 .$$

Notice that sometimes it's difficult or even impossible to eliminate the parameter t.

2)

t	x	y	(x,y)
-3	9	-2	$(9,-2)$
-2	4	-1	$(4,-1)$
-1	1	0	$(1,0)$
0	0	1	$(0,1)$
1	1	2	$(1,2)$
2	4	3	$(4,3)$
3	9	4	$(9,4)$



Remark: Notice that Cartesian Curve $x = (y-1)^2$ with $x \geq 0$ is a parabola, with

x-intercept : $(1,0)$

y-intercept : $(0,1)$.

Example: The following parametric equations describe the position $p(x,y)$ of a particle moving in the xy-plane.

(a) Identify the path traced by the particle.

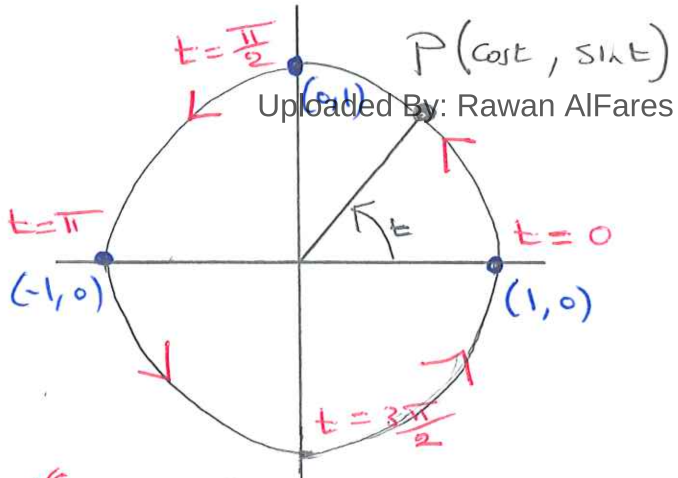
(b) Graph the parametric curve.

(c) Describe the direction of the motion.

$\boxed{1}$ $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

(a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, which is Circle centered at the origin with Radius 1.

(b) At $t = 0$, $(x,y) = (\cos 0, \sin 0) = (1,0)$ ("Initial point").



At $t = \frac{\pi}{2}$, $(x,y) = (0,1)$

At $t = \pi$, $(x,y) = (-1,0)$

At $t = \frac{3\pi}{2}$, $(x,y) = (0,-1)$

At $t = 2\pi$, $(x,y) = (1,0)$ ("terminal point")

(c) The particle moves once on the Circle $x^2+y^2=1$ Counter-clockwise starting at the point $p(1,0)$ and the arrows show the direction when t increased.

$\boxed{2}$ $x = a\cos t \quad , y = a\sin t \quad , 0 \leq t \leq 2\pi$.

(a) $x^2+y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$.

which is Circle centered at the origin with Radius a.

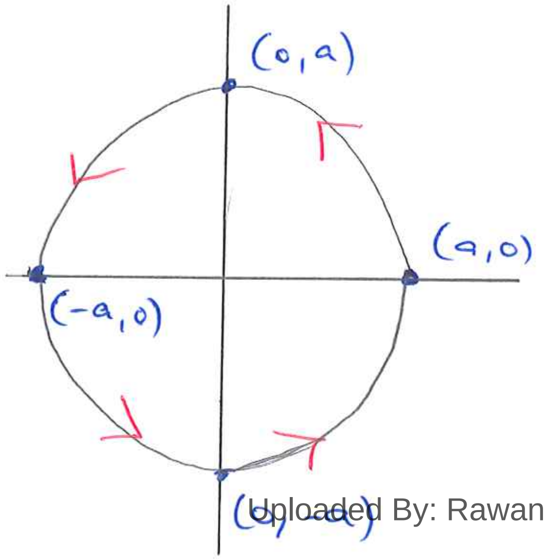
(b) at $t=0$, $(x,y) = (a,0)$

at $t=\frac{\pi}{2}$, $(x,y) = (0,a)$

at $t=\pi$, $(x,y) = (-a,0)$

at $t=\frac{3\pi}{2}$, $(x,y) = (0,-a)$

at $t=2\pi$, $(x,y) = (a,0)$



(c) The particle moves once on the Circle $x^2+y^2=a^2$ Counter Clockwise starting at the point $p(a,0)$ and the arrows show the direction when t Increased.

Remark (*): in $\boxed{1}$, if the interval of t is $0 \le t \le \pi$, then the graph of the path of the particle is the upper half of the Circle.

$\boxed{3}$ $x = 4\sin t$, $y = 5\cos t$, $0 \le t \le 2\pi$.

(a) $\frac{x}{4} = \sin t$, $\frac{y}{5} = \cos t$

$\Rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = \sin^2 t + \cos^2 t = 1$.

which is $\frac{x^2}{16} + \frac{y^2}{25} = 1$, equation of an ellipse center $(0,0)$ with major axis $x = 0$ (y-axis)

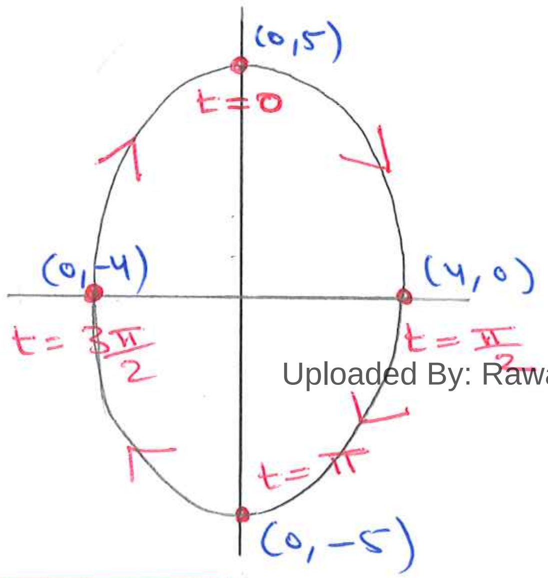
(b) At $t = 0$, $(x,y) = (0,5)$

At $t = \frac{\pi}{2}$, $(x,y) = (4,0)$

At $t = \pi$, $(x,y) = (0,-5)$

At $t = \frac{3\pi}{2}$, $(x,y) = (0,-4)$

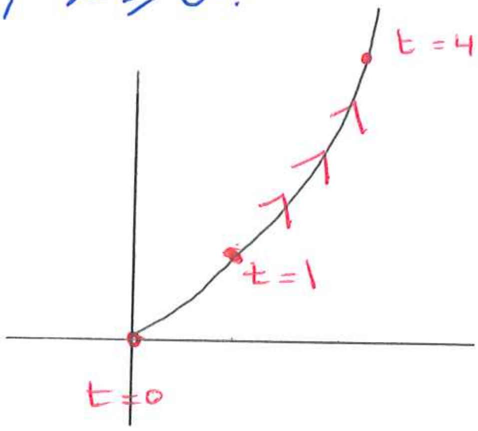
At $t = 2\pi$, $(x,y) = (0,5)$



(c) The particle Moves once on the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ Clock wise. and the arrows show the Direction as t increased.

$\boxed{4}$ $x = \sqrt{t}$, $y = t$, $t \geq 0$.

(a) $y = t = (\sqrt{t})^2 = x^2$, which is the equation of the parabola $y = x^2$, $x \geq 0$.



(b) At $t=0$, $(x,y) = (0,0)$

At $t=1$, $(x,y) = (1,1)$

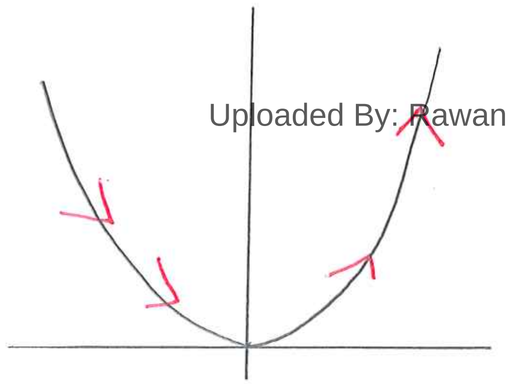
At $t=4$, $(x,y) = (2,4)$

⋮

(c) The particle moves on the parabola, with direction shown in the figure above.

$\boxed{5}$ $x = 3t$, $y = 9t^2$, $-\infty < t < \infty$

(a) $y = 9t^2 = (3t)^2 = x^2 \Rightarrow y = x^2$ which is a parabola open up.



(b) At $t=-1$, $(x,y) = (-3,9)$

At $t=0$, $(x,y) = (0,0)$

At $t=1$, $(x,y) = (3,9)$

(c) The particle moves on the parabola from left hand side of the parabola to the right.

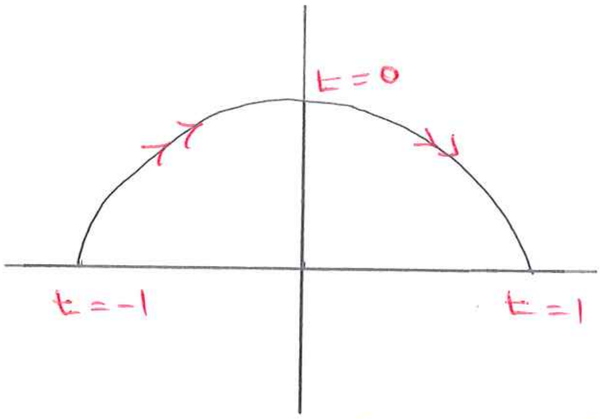
$\boxed{6}$ $x = t$, $y = \sqrt{1-t^2}$, $-1 \le t \le 1$

(a) $y = \sqrt{1-x^2} \iff x^2 + y^2 = 1$, which is the equation of a circle centered at the origin with radius $= 1$

(b) At $t = -1$, $(x,y) = (-1,0)$

At $t = 0$, $(x,y) = (0,1)$

At $t = 1$, $(x,y) = (1,0)$



(c) The particle moves on the upper half of the circle in clock wise direction.

Remark: Any curve can be represented by many different set of parametric equations (Remark (*) + $\boxed{6}$)

$\boxed{7}$ $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, $t > 0$

(a) $x^2 - y^2 = (x-y)(x+y) = \left(\left(t+\frac{1}{t}\right) - \left(t - \frac{1}{t}\right)\right)\left(\left(t+\frac{1}{t}\right) + \left(t-\frac{1}{t}\right)\right)$

$\iff x^2 - y^2 = \left(\frac{2}{t}\right)(2t) = 4 \iff x^2 - y^2 = 4.$

$\iff \frac{x^2}{2^2} - \frac{y^2}{2^2} = 1$, which is a Hyperbola with center $(0,0)$, distance between the vertices is 4 and distance between the Foci is 4, with axis $y = 0$.

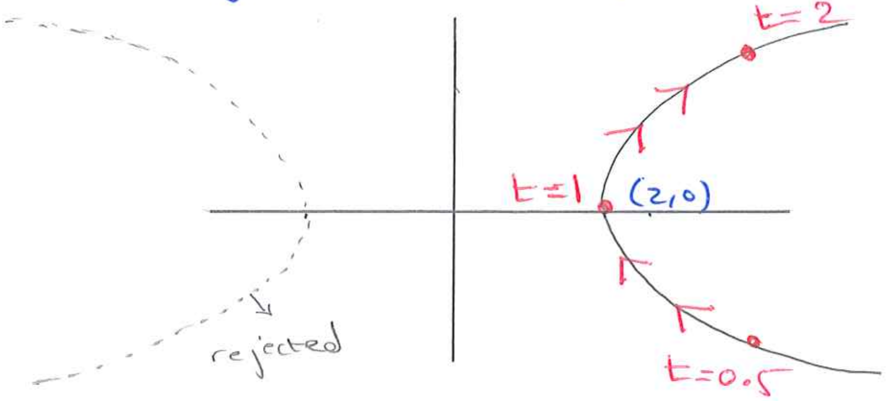
(b) At $t = 0.5$, $(x,y) = (2.5, -1.5)$.

At $t = 1$, $(x,y) = (2,0)$.

At $t = 2$, $(x,y) = (2.5, 1.5)$.

since $t > 0$

$\Rightarrow$ $x > 0$



(c) The particle moves along the Hyperbola with $x > 0$.

The arrows show the direction when t increased.

Remark: Notice that:

1). $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, $t > 0$

2). $x = \sqrt{4 + t^2}$, $y = t$, $-\infty < t < \infty$

3). $x = 2\sec t$, $y = 2\tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

are all different parametrization for the same curve (Hyperbola).

(8) (Q16) $x = -\sec t \quad , y = \tan t \quad , -\frac{\pi}{2} < t < \frac{\pi}{2}$

(a) $x^2 - y^2 = \sec^2 t - \tan^2 t = 1$, which is a Hyperbola with axis $y = 0$.

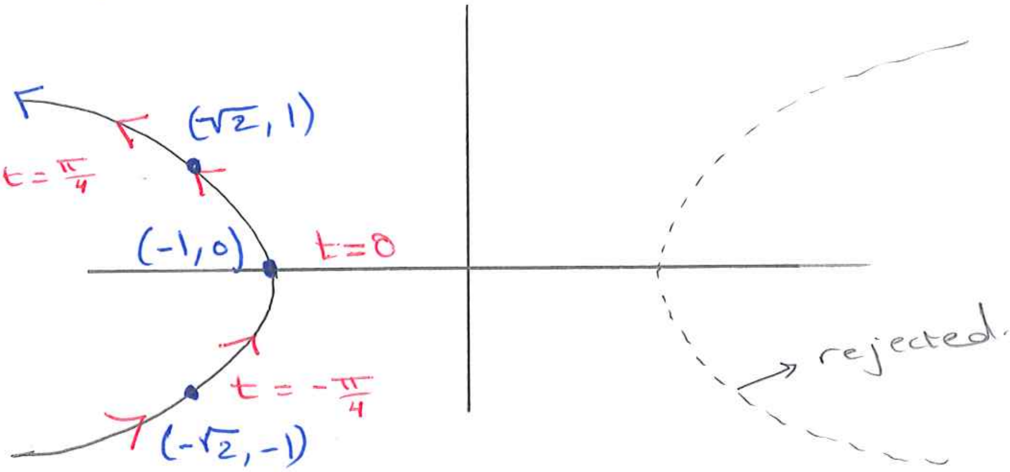
(b) As $t \to -\frac{\pi}{2}$, $x \to -\infty$ & $y \to -\infty$

At $t = -\frac{\pi}{4}$, $(x,y) = (-\sqrt{2}, -1)$

At $t = 0$, $(x,y) = (-1, 0)$

At $t = \frac{\pi}{4}$, $(x,y) = (-\sqrt{2}, 1)$

As $t \to \frac{\pi}{2}$, $x \to -\infty$, $y \to \infty$



 (c) The particle moves on the left part of the

Hyperbola. The direction is showed in the graph above.

Parametric Equation of a Line.

(a) If L is a line passes through the points:

(x_0, y_0) and (x_1, y_1), then we can parametrize

this Line as:

$$\left.\begin{aligned} x &= x_0 + (x_1 - x_0)t \\ y &= y_0 + (y_1 - y_0)t \end{aligned}\right\} , \quad -\infty < t < \infty$$

(b) If L is a line passes through the point (x_0, y_0)

with slope m, then we can parametrize this Line as

$$\left.\begin{aligned} x &= x_0 + t \\ y &= y_0 + mt \end{aligned}\right\} , \quad -\infty < t < \infty$$

(Q22)

Example: Find two different parametrization for the line segment with endpoints $(-1, 3)$ and $(3, -2)$.

① $$\left.\begin{aligned} x &= -1 + (3 - -1)t = -1 + 4t \\ y &= 3 + (-2 - 3)t = 3 - 5t \end{aligned}\right\} , \quad 0 \le t \le 1$$

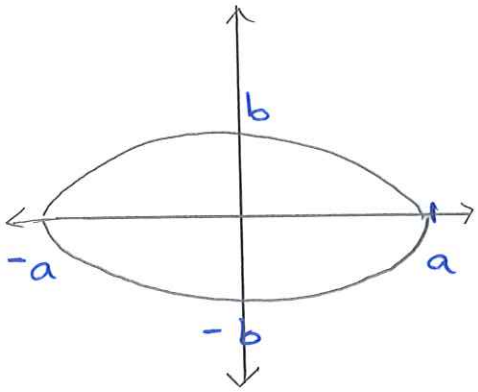
$t = 1 < \begin{matrix} x = 3 \\ y = -2 \end{matrix}$

② $m = \text{slope} = -\frac{5}{4} \Rightarrow$ $$\left.\begin{aligned} x &= -1 + t \\ y &= 3 - \frac{5}{4}t \end{aligned}\right\} , \quad 0 \le t \le 4$$

$t = 4 < \begin{matrix} x = 3 \\ y = -2 \end{matrix}$

(Q20) Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



(a) once clockwise.

$x = a\sin t$, $y = b\cos t$, $\frac{\pi}{2} \le t \le \frac{5\pi}{2}$ (دورة كاملة)

we need $\left.\begin{array}{l} a\sin t = a \\ \& \ b\cos t = 0 \end{array}\right\} \Rightarrow t = \frac{\pi}{2}$ (Initial point)

(b) once Counterclockwise

$x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$.

(c) Twice clockwise.

$x = a\sin t$, $y = b\cos t$, $\frac{\pi}{2} \le t \le \frac{9\pi}{2}$

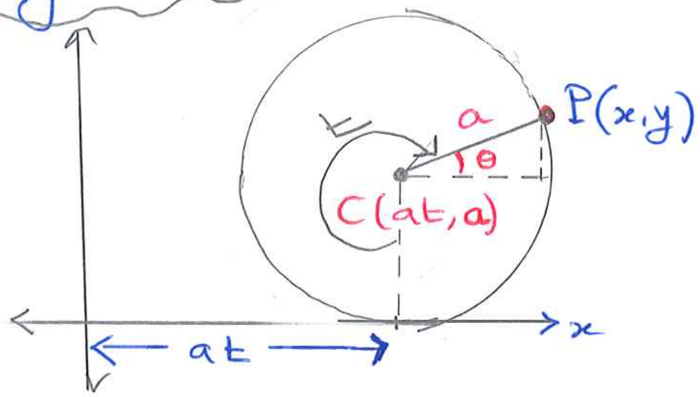
(d) Twice Counter clockwise.

$x = a\cos t$, $y = b\sin t$, $0 \le t \le 4\pi$.

Cycloids:

Example: A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference.

The path is called a Cycloid.



Sol: Let C be the center of the wheel that lies at (at, a).

And the coordinates of P are:

$$x = at + a\cos\theta \quad , \quad y = a + a\sin\theta$$

Notice that $t + \theta = \frac{3\pi}{2} \Rightarrow \theta = \frac{3\pi}{2} - t$.

$\Rightarrow \cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t$.

& $\sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t$.

Therefore, $\boxed{\begin{array}{l} x = a(t - \sin t), \\ y = a(1 - \cos t) \end{array}}$.

11.2 Calculus with Parametric Curves.

Tangents and Areas:

If $x = f(t)$ and $y = g(t)$ are differentiable functions of t, then by Chain Rule we have:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

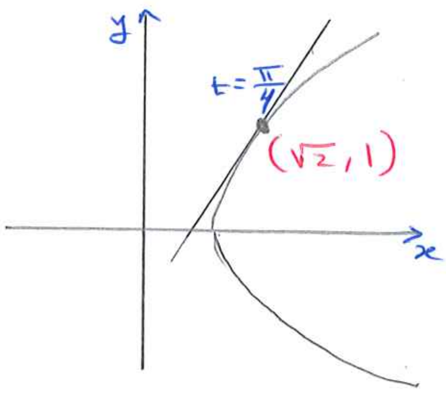
If $\frac{dx}{dt} \neq 0$, then $\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}} = y'$

Similarly, $\boxed{\frac{d^2y}{dx^2} = \frac{(dy'/dt)}{(dx/dt)}}.$

Example: Find the tangent to the curve

$$x = \sec t \;, y = \tan t \;, \; -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at $t = \frac{\pi}{4}$.



Sol: At $t = \frac{\pi}{4}$, $(x, y) = (\sqrt{2}, 1)$.

The slope of the tangent at $t = \frac{\pi}{4}$ is

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\frac{\pi}{4}} = \left.\frac{\sec^2 t}{\sec t \ \tan t}\right|_{t=\frac{\pi}{4}}$$

$$= \frac{\sec \frac{\pi}{4}}{\tan \frac{\pi}{4}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$\Rightarrow$ Tangent Line : $y - y_1 = m(x - x_1)$

$$y - 1 = \sqrt{2}\,(x - \sqrt{2})$$

$$\therefore \quad y = \sqrt{2}\, x - 1$$

Example: Find d^2y/dx^2 as a function of t

if $x = t - t^2$, $y = t - t^3$.

sol $y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^2}$$

$$= \frac{6t^2 - 6t + 2}{(1-2t)^2}.$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{6t^2-6t+2}{(1-2t)^2} \cdot \frac{1}{(1-2t)}$$

$$= \frac{6t^2-6t+2}{(1-2t)^3}.$$

Example: Find the Normal to the Curve:

$$x = 2t^2+3 \ , \ y = t^4 \quad \text{at } t=-1$$

sol: The slope of the tangent to the curve at $t=-1$ is

$$m_T = \left.\frac{dy}{dx}\right|_{t=-1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=-1} = \left.\frac{4t^3}{4t}\right|_{t=-1} = 1.$$

then slope of the normal line $= \frac{-1}{m_T} = \boxed{-1}$.

Moreover, at $t=-1$, $(x,y) = (5,1)$

$\therefore$ Equation of the Normal is

$$y-1 = -1(x-5)$$

$$\Rightarrow \ y = 6-x$$

Example: Find the Slope of the Curve:

$$x^3 + 2t^2 = 9 \ , \quad 2y^3 - 3t^2 = 4 \ , \quad \text{at } t = 2.$$

Sol: At $t = 2$:

$$x^3 + 2(2)^2 = 9 \Rightarrow x^3 = 1 \Rightarrow \boxed{x = 1}$$

$$2y^3 - 3(2)^2 = 4 \Rightarrow 2y^3 = 16 \Rightarrow \boxed{y = 2}$$

Now, $m = \dfrac{dy/dt}{dx/dt}$,

$$3x^2 \cdot \frac{dx}{dt} + 4t = 0 \Rightarrow \frac{dx}{dt} = \frac{-4t}{3x^2}$$

& $$6y^2 - 6t = 0 \Rightarrow \frac{dy}{dt} = \frac{6t}{6y^2} = \frac{t}{y^2}.$$

$$\therefore m = \left.\frac{t/y^2}{-4t/3x^2}\right|_{t=2} = \frac{2/(2)^2}{-4(2)/3(1)^2} = \frac{1}{2} \cdot \left(\frac{-3}{8}\right)$$

$$= \boxed{\frac{-3}{16}}.$$

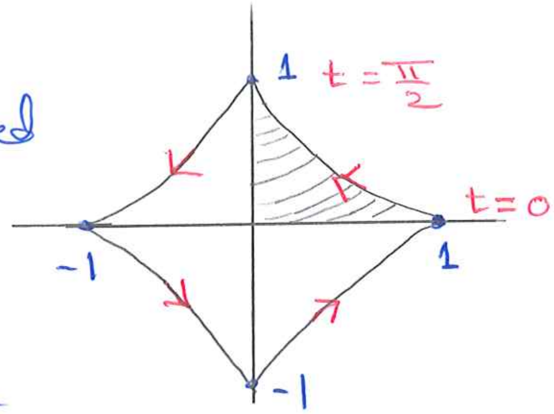
Area : If $a \le x \le b$ & $y = f(x) \ge 0$

then $\text{Area} = \int_a^b f(x)\,dx$.

Like a square with concave sides.

Example: Find the area enclosed by the astroid

$$x = \cos^3 t \quad , \quad y = \sin^3 t \quad , \quad 0 \le t \le 2\pi.$$



sol: By symmetry, the enclosed area is 4 times the area beneath the curve in the first quadrant where $0 \le t \le \frac{\pi}{2}$.

$$A = 4\int_0^1 y\,dx$$

when $x = 0 \Rightarrow 0 = \cos^3 t \Rightarrow t = \frac{\pi}{2}$

when $x = 1 \Rightarrow 1 = \cos^3 t \Rightarrow t = 0$

$$\therefore A = 4\int_0^1 y\,dx = 4\int_{\frac{\pi}{2}}^{0} \sin^3 t\,(3\cos^2 t)(-\sin t)\,dt.$$

$$= 12\int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t\,dt$$

$$= 12\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2t}{2}\right)^2\left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1-2\cos 2t+\cos^2 2t)(1+\cos 2t)\, dt.$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2t-\cos^2 2t+\cos^3 2t)\, dt$$

$$= \frac{3}{2}\left[\underbrace{\int_0^{\frac{\pi}{2}} (1-\cos 2t)\,dt}_{I_1} - \underbrace{\int_0^{\frac{\pi}{2}} \cos^2 2t\, dt}_{I_2} + \underbrace{\int_0^{\frac{\pi}{2}} \cos^3 2t\, dt}_{I_3}\right]$$

$$I_1 = \int_0^{\frac{\pi}{2}} (1-\cos 2t)\, dt = t - \frac{\sin 2t}{2}\Big|_0^{\frac{\pi}{2}} = \boxed{\frac{\pi}{2}}$$

$$I_2 = \int_0^{\frac{\pi}{2}} \cos^2 2t\, dt = \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 4t}{2}\right) dt = \frac{1}{2}\left[t+\frac{\sin 4t}{4}\right]\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+0-0\right] = \boxed{\frac{\pi}{4}}$$

$$I_3 = \int_0^{\frac{\pi}{2}} \cos^3 2t\, dt = \int_0^{\frac{\pi}{2}} \cos^2 2t \cos 2t\, dt = \int_0^{\frac{\pi}{2}} (1-\sin^2 2t)\cos 2t\, dt$$

$$= \int_0^0 (1-u^2)\frac{du}{2} = \boxed{0}$$

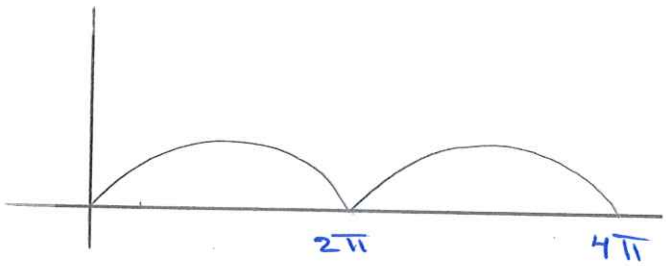
$$\therefore A = \frac{3}{2}\left[I_1 - I_2 + I_3\right] = \frac{3}{2}\left[\frac{\pi}{2}-\frac{\pi}{4}+0\right] = \boxed{\frac{3\pi}{8}} \ (161)$$

Example: Find the area under one arch of the cycloid:

$$x = a(t - \sin t) \ , \ y = a(1 - \cos t) \quad \text{when } a = 1$$

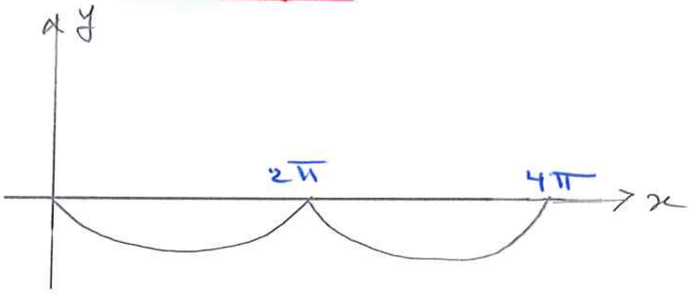
Sol: $A = \int_0^{2\pi} y \, dx = \int_0^{2\pi} (1 - \cos t)(1 - \cos t) \, dt$

$$= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) \, dt = \dots = 3\pi$$

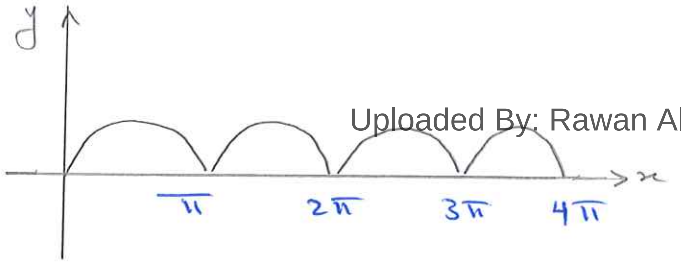


$a = 1$

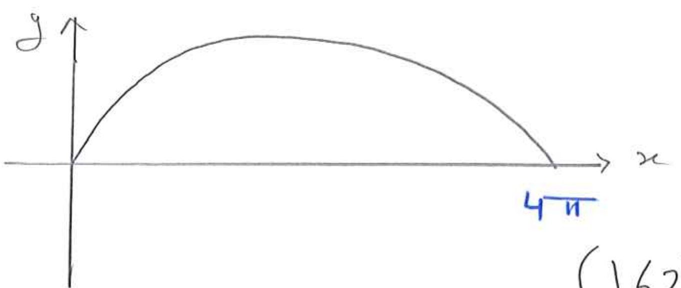
Remark: When $a = -1$



When $a = \frac{1}{2}$



When $a = 2$



Example: (Q22) Find the Area enclosed by the y-axis and the Curve $x = t - t^2$, $y = 1 + e^{-t}$, $0 \le t \le 1$

sol: $A = \int_0^1 x \, dy = \int_0^1 (t - t^2)(-e^{-t}) \, dt = -\int_0^1 (t - t^2) e^{-t} \, dt$

$$= -\left[(t-t^2)(-e^{-t}) - (1-2t)e^{-t} + 2e^{-t} \right] \Big|_0^1$$

$t - t^2$	$+$	e^{-t}
$1 - 2t$	$-$	$-e^{-t}$
-2	$+$	e^{-t}
0		$-e^{-t}$

$$= (t-t^2)e^{-t} + (1-2t)e^{-t} - 2e^{-t} \Big|_0^1$$

$$= (0 - e^{-1} - 2e^{-1}) - (0 + 1 - 2) = -\frac{3}{e} + 1.$$

Length of Parametrically defined Curve.

Def: If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are Continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then the length of C is:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt.$$

Remark : The length of a Curve $y = f(x)$ is a special Case of equation $\boxed{**}$. Given a Continuously differentiable function $y = f(x)$, $a \le x \le b$, we can assign $x = t$ as a parameter. The graph of the function f is then the Curve C defined parametrically by :

$$x = t \quad , \quad y = f(t) \quad , \quad a \le t \le b$$

$$\Rightarrow \quad \frac{dx}{dt} = 1 \quad \& \quad \frac{dy}{dt} = f'(t)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = f'(t)$$

Therefore, $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 + \left(f'(t)\right)^2$. $\boxed{x = t}$

$$\Rightarrow \quad L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Example: Find the Length of the Circle of radius r defined by:

$$x = r\cos t \quad , \quad y = r\sin t \quad , \quad 0 \le t \le 2\pi .$$

sol: $$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2}\, dt = r\int_0^{2\pi} 1\, dt = \boxed{2\pi r}$$

Example: Find the length of the astroid

$$x = \cos^3 t \quad , \quad y = \sin^3 t \quad , \quad 0 \le t \le 2\pi$$

sol: $$\frac{dx}{dt} = (3\cos^2 t)(-\sin t) = -3\sin t\cos^2 t .$$

$$\frac{dy}{dt} = 3\sin^2 t\cos t .$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^2 t\cos^4 t + 9\sin^4 t\cos^2 t$$

$$= 9\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)$$

$$= 9\sin^2 t\cos^2 t .$$

$$\therefore \; L = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^2 t \sin^2 t}\; dt$$

$$= 4\int_0^{\frac{\pi}{2}} |3\sin t \cos t|\; dt$$ ≥ 0 (First quadrant).

$$= 4\int_0^{\frac{\pi}{2}} \frac{3}{2}\sin 2t\; dt = 6\left(-\frac{\cos 2t}{2}\right)\Bigg|_0^{\frac{\pi}{2}} = \boxed{6}$$

Areas of Surfaces of Revolution.

Area of Surface of Revolution for Parametrized Curves.

If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, is traversed exactly once as t increases from a to b, then the area of the surfaces generated by revolving the curve about the coordinate axes are as follows:

1. Revolution about the x-axis $(y \geq 0)$:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

2. Revolution about the y-axis $(x \geq 0)$:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example: Find the area of the surface generated by revolving the circle

$x = \cos t$, $y = 1 + \sin t$, $0 \leq t \leq 2\pi$

about x-axis.

Sol: Notice that the circle has radius 1 & center $(0,1)$.

$$S = \int_0^{2\pi} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} 2\pi (1+\sin t) \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$$= 2\pi \int_0^{2\pi} (1+\sin t)\, dt = \cdots = \boxed{4\pi^2}$$

Example: (Q34) Find the area of the Surface generated by revolving the Curve

$x = \ln(\sec t + \tan t) - \sin t$, $y = \cos t$,

$0 \le t \le \frac{\pi}{3}$ about x-axis.

Sol:

check !!

$$\frac{dx}{dt} = \frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t} - \cos t = \frac{\sin^2 t}{\cos t}$$

$$\frac{dy}{dt} = -\sin t$$

$$\therefore S = 2\pi \int_0^{\frac{\pi}{3}} \cos t \sqrt{\frac{\sin^4 t}{\cos^2 t} + \sin^2 t} \; dt$$

$$= 2\pi \int_0^{\frac{\pi}{3}} \cos t \sqrt{\frac{\sin^4 t + \sin^2 t \cos^2 t}{\cos^2 t}} \; dt$$

$$= 2\pi \int_0^{\frac{\pi}{3}} \cos t \sqrt{\frac{\sin^2 t(\sin^2 t + \cos^2 t)}{\cos^2 t}} \; dt$$

$$= 2\pi \int_0^{\frac{\pi}{3}} \cancel{\cos t} \cdot \frac{\sin t}{\cancel{\cos t}} \, dt = -2\pi \cos t \Big|_0^{\frac{\pi}{3}} = \boxed{\pi}.$$