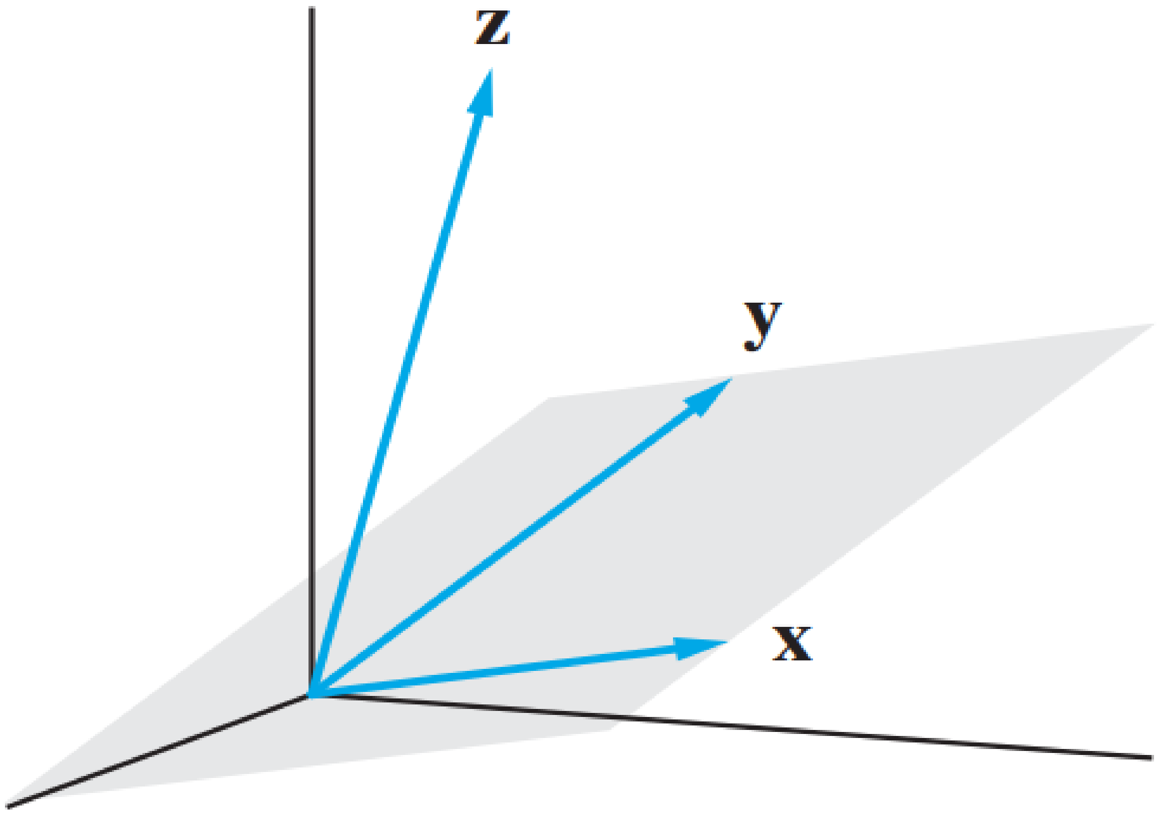
CHAPTER

3



Vector Spaces

Vectors in n-space R^n (basics)

- The basics about n-vectors are straight forward generalization of what you have thoroughly covered in earlier courses.
- We will quickly refresh the material you already know without going into much detail.

- **n-vectors**

  Given by n-tuples $\langle a_1, a_2, \cdots, a_n \rangle$

  with each a_i a real number

  $a_1, a_2, \cdots, a_n$ are called components of vector

- **Addition/Subtraction of n-vectors**

$$\langle a_1, a_2, \cdots, a_n \rangle \pm \langle b_1, b_2, \cdots, b_n \rangle = \langle a_1 \pm b_1, a_2 \pm b_2, \cdots, a_n \pm b_n \rangle$$

component-wise addition/subtraction

- **Multiplication by a scalar (i.e. by a real number)**

Given a real number k . Then

$$k\langle a_1, a_2, \cdots, a_n \rangle = \langle ka_1, ka_2, \cdots, ka_n \rangle$$

multiplying each component

Vector space

- Here we study those sets of vectors which have special properties and play important role in applications.

Definition

Let V be a set on which the operations of addition and scalar multiplication are defined. By this we mean that, with each pair of elements $\mathbf{x}$ and $\mathbf{y}$ in V, we can associate a unique element $\mathbf{x}+\mathbf{y}$ that is also in V, and with each element $\mathbf{x}$ in V and each scalar α, we can associate a unique element $\alpha\mathbf{x}$ in V. The set V together with the operations of addition and scalar multiplication is said to form a **vector space** if the following axioms are satisfied:

A1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for any $\mathbf{x}$ and $\mathbf{y}$ in V.

A2. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ for any $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in V.

A3. There exists an element $\mathbf{0}$ in V such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for each $\mathbf{x}\in V$.

A4. For each $\mathbf{x}\in V$, there exists an element $-\mathbf{x}$ in V such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.

A5. $\alpha(\mathbf{x}+\mathbf{y})=\alpha\mathbf{x}+\alpha\mathbf{y}$ for each scalar α and any $\mathbf{x}$ and $\mathbf{y}$ in V.

A6. $(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}$ for any scalars α and β and any $\mathbf{x}\in V$.

A7. $(\alpha\beta)\mathbf{x}=\alpha(\beta\mathbf{x})$ for any scalars α and β and any $\mathbf{x}\in V$.

A8. $1\mathbf{x}=\mathbf{x}$ for all $\mathbf{x}\in V$.

Example. Let $V = \mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$. If, $\mathbf{x}, \mathbf{y} \in V$, that is if, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, define the vector space operations on V by:

$$\mathbf{x} \oplus \mathbf{y} = (x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$k \odot \mathbf{x} = k \odot (x_1, x_2) = (kx_1, kx_2)$$

Then V with the operations $\oplus$ and $\odot$ is a vector space.

Remark:

We leave it to the reader to verify that $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$, with the usual addition and scalar multiplication of matrices, are both vector spaces. There are a number of other important examples of vector spaces.

An important component of the definition is the closure properties of the two operations. These properties can be summarized as follows:

C1. If $\mathbf{x} \in V$ and α is a scalar, then $\alpha\mathbf{x} \in V$.

C2. If $\mathbf{x}, \mathbf{y} \in V$, then $\mathbf{x} + \mathbf{y} \in V$.

To illustrate the necessity of the closure properties, consider the following example: Let

$$W = \{(a, 1) \mid a \text{ real}\}$$

with addition and scalar multiplication defined in the usual way. The elements $(3, 1)$ and $(5, 1)$ are in W, but the sum

$$(3, 1) + (5, 1) = (8, 2)$$

is not an element of W. The operation $+$ is not really an operation on the set W because property C2 fails to hold. Similarly, scalar multiplication is not defined on W, because property C1 fails to hold. The set W, together with the operations of addition and scalar multiplication, is *not* a vector space.

Example. Let $V = \mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$. If, $\mathbf{x}, \mathbf{y} \in V$, that is if, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, define the vector space operations on V by:

$$\mathbf{x} \oplus \mathbf{y} = (x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$k \odot \mathbf{x} = k \odot (x_1, x_2) = (kx_1, 0)$$

Show that V with the operations $\oplus$ and $\odot$ is not a vector space.

Example. Let $V = F(-\infty, \infty)$ be the set of all real-valued functions that are defined at each x in the interval $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in V and if k is any scalar, then define the vector space operations on V by:

$$(\mathbf{f} \oplus \mathbf{g})(x) = f(x) + g(x)$$
$$(k \odot \mathbf{f})(x) = kf(x)$$

Show that V with the operations $\oplus$ and $\odot$ is a vector space.

The Vector Space $C[a, b]$

Let $C[a, b]$ denote the set of all real-valued functions that are defined and continuous on the closed interval $[a, b]$. In this case, our universal set is a set of functions. Thus, our vectors are the functions in $C[a, b]$. The sum $f + g$ of two functions in $C[a, b]$ is defined by

$$(f + g)(x) = f(x) + g(x)$$

for all x in $[a, b]$. The new function $f + g$ is an element of $C[a, b]$ since the sum of two continuous functions is continuous. If f is a function in $C[a, b]$ and α is a real number, define αf by

$$(\alpha f)(x) = \alpha f(x)$$

for all x in $[a, b]$. Clearly, αf is in $C[a, b]$ since a constant times a continuous function is always continuous.

We leave it to the reader to verify that the remaining vector space axioms are all satisfied.

The Vector Space P_n

Let P_n denote the set of all polynomials of degree less than n. Define $p + q$ and αp, respectively, by

$$(p + q)(x) = p(x) + q(x)$$

and

$$(\alpha p)(x) = \alpha p(x)$$

for all real numbers x. In this case, the zero vector is the zero polynomial,

$$z(x) = 0x^{n-1} + 0x^{n-2} + \cdots + 0x + 0$$

It is easily verified that all the vector space axioms hold. Thus, P_n, with the standard addition and scalar multiplication of functions, is a vector space.

Theorem 3.1.1 *If V is a vector space and* $\mathbf{x}$ *is any element of V, then*

(i) $0\mathbf{x} = \mathbf{0}$.
(ii) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ *implies that* $\mathbf{y} = -\mathbf{x}$ *(i.e., the additive inverse of* $\mathbf{x}$ *is unique).*
(iii) $(-1)\mathbf{x} = -\mathbf{x}$.

Proof It follows from axioms A6 and A8 that

$$\mathbf{x} = 1\mathbf{x} = (1 + 0)\mathbf{x} = 1\mathbf{x} + 0\mathbf{x} = \mathbf{x} + 0\mathbf{x}$$

Thus

$$-\mathbf{x} + \mathbf{x} = -\mathbf{x} + (\mathbf{x} + 0\mathbf{x}) = (-\mathbf{x} + \mathbf{x}) + 0\mathbf{x} \qquad \text{(A2)}$$
$$\mathbf{0} = \mathbf{0} + 0\mathbf{x} = 0\mathbf{x} \qquad \text{(A1, A3, and A4)}$$

To prove (ii), suppose that $\mathbf{x} + \mathbf{y} = \mathbf{0}$. Then

$$-\mathbf{x} = -\mathbf{x} + \mathbf{0} = -\mathbf{x} + (\mathbf{x} + \mathbf{y})$$

Therefore,

$$-\mathbf{x} = (-\mathbf{x} + \mathbf{x}) + \mathbf{y} = \mathbf{0} + \mathbf{y} = \mathbf{y} \qquad \text{(A1, A2, A3, and A4)}$$

Finally, to prove (iii), note that

$$\mathbf{0} = 0\mathbf{x} = (1 + (-1))\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x} \qquad \text{[(i) and A6]}$$

Thus

$$\mathbf{x} + (-1)\mathbf{x} = \mathbf{0} \qquad \text{(A8)}$$

and it follows from part (ii) that

$$(-1)\mathbf{x} = -\mathbf{x}$$