Chapter (2)

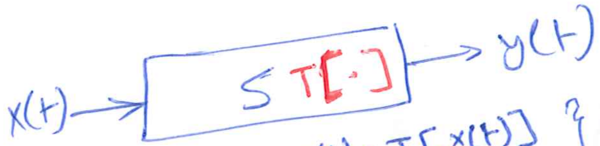
System modeling and analysis in the time domain:-

System:- a combination and interconnection of several components to perform a task.

Modeling:- a set of mathematical equations used to represent a physical system relating the system output to its input

Classification aspects

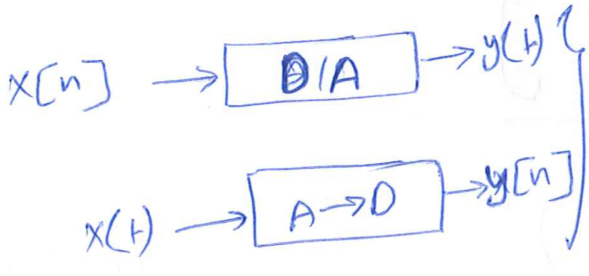
- system type (electrical, electronics, mechanical, Hydrolic, ...)
- system categories (open, closed)
- system operation (CD, DD)



continuous time system

$y(t) = T[x(t)]$
$= S[x(t)]$ } Transformation, produces, or leads to

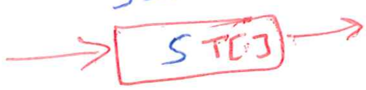
discrete time system



Hybrid system
Digital to Analog converter
Analog to Digital converter

- number of variables

SISO



MIMO

- System characteristics

System characteristics
- → Memory, Memoryless
  dynamic, static, instantaneous
- → Causal, non Causal
- → linear, non linear
- → Time Variant, Time inveriant
- ~~invertible, non invertible~~ (Fixed)
- → stable, non stable
- → invertible, non invertable

1-) memory, memory less

The system is dynamic (memory) if the output at any time depends on the input at that time and past or future value

The system is static (Memoryless) if the o/p $y(t)$ at $t=t_0$ depends only on i/p at $t=t_0$

$y(t) = x(t)$
$y(t) = x(t) + A$
$y(t) = A\,x(t)$
$y(t) = x(t) + \sin(3t)$
$y(t) = x^2(t)$
$y[n] = x[n] + x^2[n]$

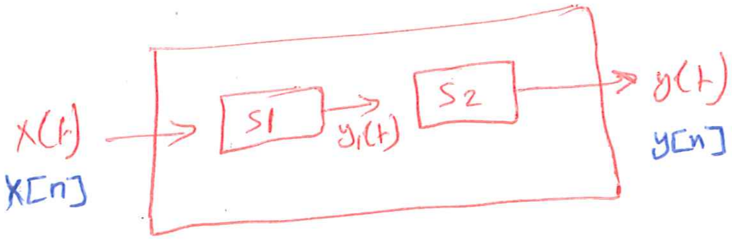
memoryless
static

$y(t) = x(t-2)\ ,\ x(t+3)$
$y(t) = x(At)$
$y(t) = x(t^2)$
$y[n] = \sum_{n=1}^{\infty} x[n]$
$y(t) = \int_{-\infty}^{t} x(t)$

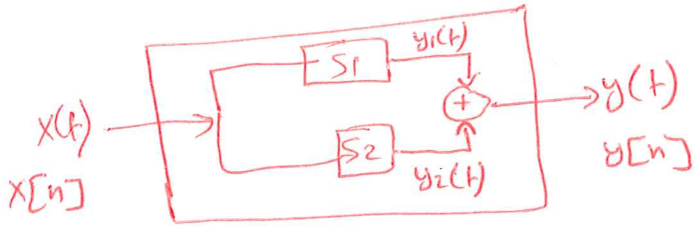
memory
Dynamic

System Connection

Complex System ⟶ subsystems



① - Series (Cascade) connection

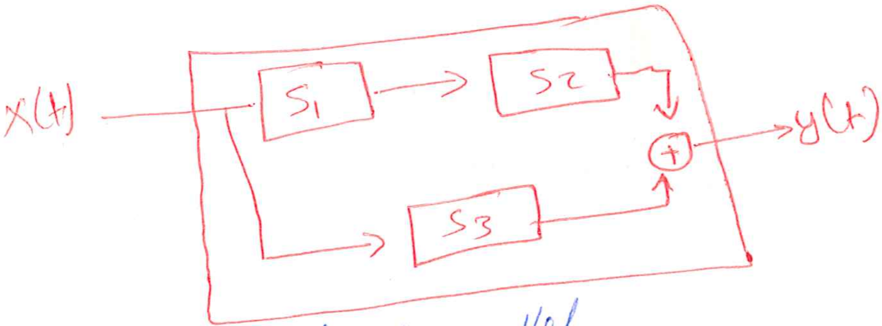


② Parallel connection

⊕ summing junction

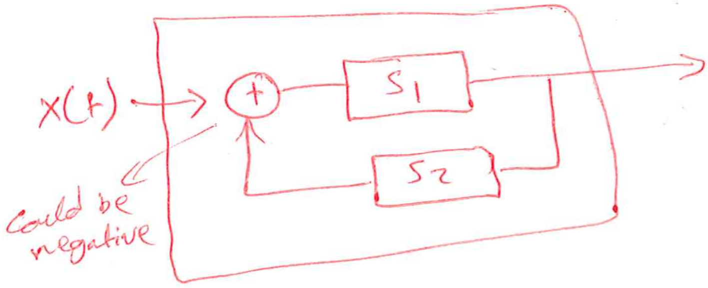
ⓧ Product junction

 scaling junction



series & parallel

③ feedback connection



2- Causal and non Causal Systems:

Causal: The output at any time depends on the instant value of the input or at past value

non Causal: The o/p depends on future value of the I/p

all real time systems are Causal systems.

$$y(t) = x(t) - x(t-1)$$

$$y[n] = \sum_{n=-\infty}^{n} x[n]$$

Causal

$$y[n] = \sum_{n=-\infty}^{n+4} x[n]$$

$$y(t) = x(2t)$$

$$y(t) = x(t^2)$$

$$y(t) = x(-t)$$

$$y(t) = x(t) + 2x(3-t)$$

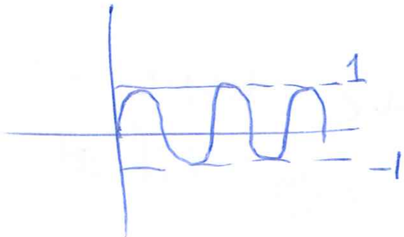
non Causal

3- stable and non stable system

The system is stable if bounded input leads to bounded output (BIBO) and unstable if bounded input leads to unbounded output

The signal $x(t)$ is bounded if $|x(t)| \le M$, $\forall t$

$\sin(t)$



$y(t) = \sin(x(t))$ $\longleftarrow$ bounded $y(t) = \sin(M)$

$y(t) = e^{x(t)}$ $\longleftarrow$ bounded $y(t) = e^{M}$

$y(t) = \int_{-\infty}^{t} x(t)\, dt$ $\longleftarrow$ unbounded $y(t) = t$

$y(t) = t\, x(t)$

4- Invertible and non Invertible system

$x(t) \xrightarrow{\text{leads}} y(t)$

Then the system is Invertible if $y(t)$ can be used to give $x(t)$ back.

$y(t) = K x(t)$

$x(t) = \dfrac{y(t)}{K}$ invertible

$y(t) = 0$, A?

$y(t) = x^2(t)$ non invertible

5- Time Variant and time inveriant system

The system said to be time inveriant

If $x(t) \xrightarrow{\text{leads to}} y(t)$ the $x(t-t_0) \longrightarrow y(t-t_0)$

- If you did not change the shape of the input, you just shift the input to the left or to the right then $\Rightarrow$ the shape of the output will be the same and only shifted to the left or to the right

① $y(t) = \sin(x(t)) \longrightarrow$

$x_1(t) \longrightarrow y_1(t) = \sin(x_1(t))$

$x_2(t) = x_1(t-t_0) \Rightarrow y_2(t) = \sin(x_2(t))$

$= \sin(x_1(t-t_0))$

now shift $y_1(t)$

$y_1(t-t_0) = \sin(x_1(t-t_0))$

if $y_2 = y_1$ then the system is time inveriant

$y(t) = \sin(t)\, x(t)$

$x_1(t) \longrightarrow y_1(t) = \sin(t)\, x_1(t)$

$x_2(t) = x_1(t-t_0) \longrightarrow y_2(t) = \sin(t)\, x_2(t)$

$= \sin(t)\, x_1(t-t_0)$

$y_1(t-t_0) = \sin(t-t_0)\, x_1(t-t_0)$

$y_1(t-t_0) \neq y_2(t) \longrightarrow$ time Variant System

$y[n] = n\, x[n]$

$x_1[n] \longrightarrow y_1 = n\, x_1[n]$

$x_2[n] = x_1[n-n_0] \longrightarrow y_2[n] = n\, x_2[n]$

$= n\, x_1[n-n_0]$

$y_1[n-n_0] = (n-n_0)\, x_1[n-n_0]$

$y_1[n-n_0] \neq y_2[n] \Rightarrow$ Time Variant

6- Linear and Non linear systems

most of practical systems are not linear but we can linearize them.

The system (S) is linear if it satisfies principles of

1) Superposition

$$x_1(t) \longrightarrow y_1(t)$$
$$x_2(t) \longrightarrow y_2(t)$$
$$x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$$

2) Homogeniety

$$\alpha x_1(t) \longrightarrow \alpha y_1(t)$$

zero input $\longrightarrow$ zero output

To check the system Linearity

1- assume two inputs to the system
$x_1(t)$, $x_2(t)$

2- find the output to each input
$y_1(t)$ and $y_2(t)$

3- find the weighted sum of the outputs
$y_3(t) = \alpha y_1 + \alpha y_2$

4- find the output due to weighted sum of inputs
$y_4(t) = f(\underbrace{\alpha x_1(t) + \alpha x_2(t)}_{x_4(t)})$

5- if $y_3(t) = y_4(t) \longrightarrow$ the system is linear, if not, the system is not linear.

Example:

1) $y(t) = A\,x(t)$ Check the system linearity:

$y_1(t) = A\,x_1(t)$

$y_2(t) = A\,x_2(t)$

$y_3(t) = \alpha A\,x_1(t) + \alpha A\,x_2(t)$

$y_4(t) = A[\alpha x_1(t) + \alpha x_2(t)]$

$= \alpha A\,x_1(t) + \alpha A\,x_2(t)$

$y_3(t) = y_4(t) \Rightarrow$ the system is linear

2) $y(t) = m\,x(t) + C$

$y_1(t) = m\,x_1(t) + C$

$y_2(t) = m\,x_2(t) + C$

$y_3(t) = \alpha m\,x_1(t) + \alpha C + \alpha m\,x_2(t) + \alpha C$

$y_4(t) = m[\alpha x_1(t) + \alpha x_2(t)] + C$

$= \alpha m\,x_1(t) + \alpha m\,x_2(t) + C$

$y_3(t) \neq y_4(t)$ $\therefore$ the system is not linear

4) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$$y_1(t) = \int_{-\infty}^{t} x_1(\tau)\, d\tau$$

$$y_2(t) = \int_{-\infty}^{t} x_2(\tau)\, d\tau$$

$$y_3(t) = \alpha \int_{-\infty}^{t} x_1(\tau)\, d\tau + \alpha \int_{-\infty}^{t} x_2(\tau)\, d\tau$$

$$y_4(t) = \int_{-\infty}^{t} [\alpha x_1(\tau) + \alpha x_2(\tau)]\, d\tau$$

$$= \int_{-\infty}^{t} \alpha x_1(\tau)\, d\tau + \int_{-\infty}^{t} \alpha x_2(\tau)\, d\tau$$

$y_3(t) = y_4(t) \Rightarrow$ the system is linear

5) $y[n] = 10 x^2[n]$

$$y_1[n] = 10\, x_1^2[n]$$

$$y_2[n] = 10\, x_2^2[n]$$

$$y_3[n] = \alpha 10\, x_1^2[n] + \alpha 10\, x_2^2[n]$$

$$y_4[n] = 10 \left[\alpha x_1[n] + \alpha x_2[n]\right]^2$$

$$= 10 \alpha^2 x_1^2[n] + 20\alpha^2 x_1[n] x_2[n] + 10\alpha^2 x_2^2[n]$$

$y_3[n] \neq y_4[n] \Rightarrow$ the system is not linear

5-
$$\frac{dy(t)}{dt} + 10y(t) + 5 = x(t)$$

$$\frac{dy_1(t)}{dt} + 10y_1(t) + 5 = x_1(t)$$

$$\frac{dy_2(t)}{dt} + 10y_2(t) + 5 = x_2(t)$$

$$y_3(t) = \alpha\frac{dy_1(t)}{dt} + \alpha 10y_1(t) + \alpha 5 + \alpha\frac{dy_2(t)}{dt} + \alpha 10 y_2(t) + 5\alpha$$

$$= \alpha\frac{d}{dt}[y_1(t)+y_2(t)] + \alpha 10[y_1(t)+y_2(t)] + \alpha 10 \quad = \alpha x_1(t) + \alpha x_2(t) = \alpha[x_1(t) + x_2(t)]$$

$$y_4(t) = \alpha\frac{d}{dt}[y_1(t)+y_2(t)] + 10\alpha[y_1(t)+y_2(t)] = \alpha[x_1(t)+x_2(t)]$$

$y_3(t) \neq y_4(t) \Longrightarrow$ non linear system

If the constant (5) was not there then the system will be linear

* Continuouse linear time inveriant system

LTI system

most of the practical systems could be linearized and explained independent of time.

now we can mathematically represent (model) any complex signal as a sum of basic signals

$$x(t) = x_1(t) + x_2(t) + \dots$$

and acording to the linearity property

$$y(t) = y_1(t) + y_2(t) + \dots$$

$$\alpha x_1(t) \longrightarrow \alpha y_1(t)$$

and acording to the time inveriant property

$$x(t-t_0) \longrightarrow y(t-t_0)$$

* Impulse Response of LTI system

$$\delta(t) \longrightarrow \boxed{LTI} \xrightarrow{??} y(t)$$

$$\downarrow$$

$$h(t)$$

$$\rightarrow \quad x(t) \longrightarrow y(t)$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\delta(at) = \frac{\delta(t)}{|a|}$$

$$\delta(-t) = \delta(t)$$

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

$$\int_{t_1}^{t_2} x(t)\,\delta(t-t_0)\,dt = \begin{cases} x(t_0) & t_1 \le t_0 \le t \\ 0 & \text{o.w} \end{cases}$$

$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t)$$

$$\delta(t-\tau) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t-\tau)$$

Time inveriant

$$x(\tau)d\tau\,\delta(t-\tau) \longrightarrow \boxed{\text{LTI}} \longrightarrow x(\tau)\,d\tau\,h(t-\tau)$$

Linearity

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow \boxed{\text{LTI}} \longrightarrow \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$\downarrow$$

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau$$

① $x(t) * h(t) = h(t) * x(t)$

Convolution is Commutative تبادلي

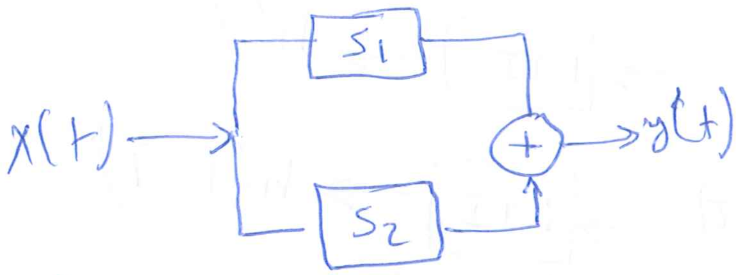
② $(x(t) * h_1(t)) * h_2(t) = x(t) * (h_1(t) * h_2(t))$

Associative

3- $x(t) * [h_1(t) + h_2(t)]$

$= x(t) * h_1(t) + x(t)\, h_2(t)$

Distributive



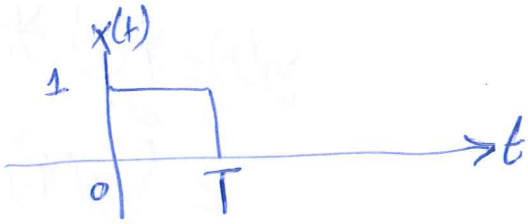
Evaluation of Convolution:

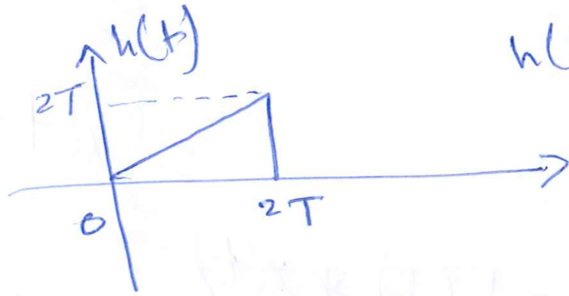
① Graphically → most useful technique

② Analytically

Examples:

$x(t) = u(t) - u(t-T)$





$h(t) = u(2T - t)\, u(t)$

$= [u(t) - u(t-2T)]\, t$

$y(t) = x(t) * h(t)$

$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)$

-1- plot $x(\tau)$ Vs τ

-2- plot $h(\tau)$ Vs τ

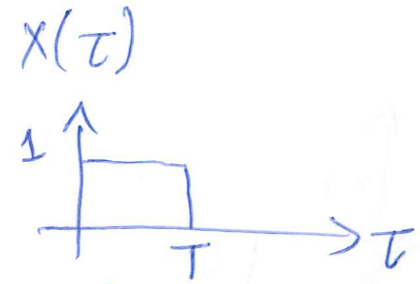
-3- plot $h(-\tau)$ Vs τ

-4- Plot $h(t-\tau)$ Vs τ

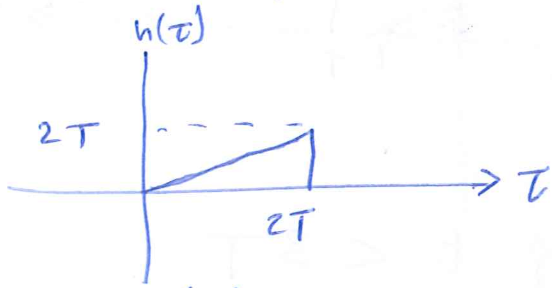
constant variable

-5- find the area under the multiplication $= y(t)$

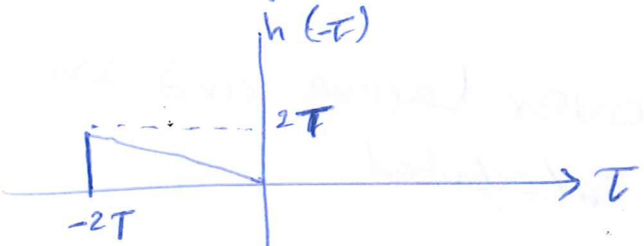
1-



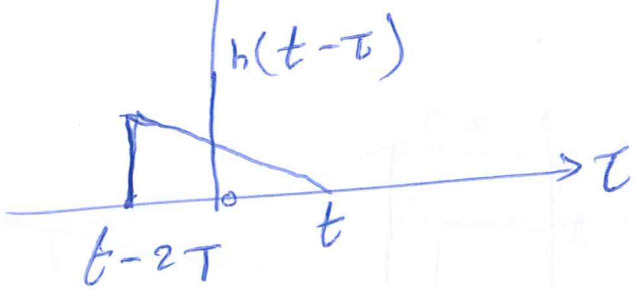
2-



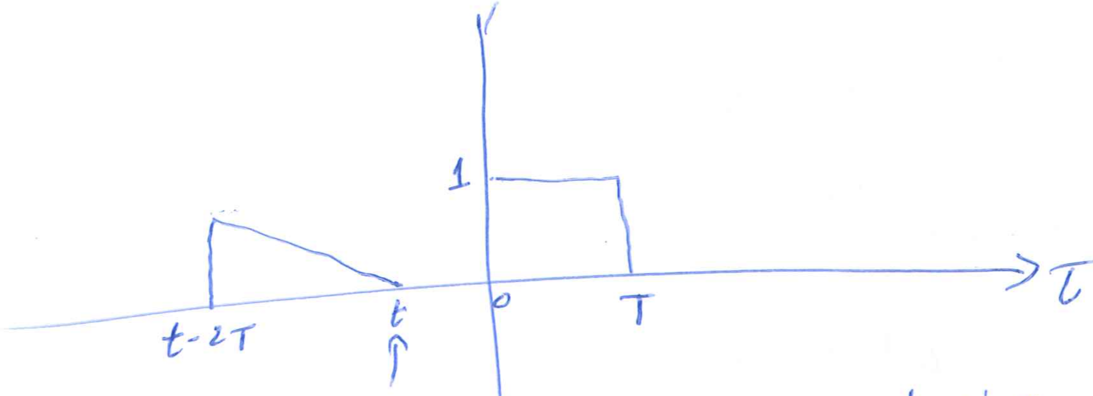
3-



4)



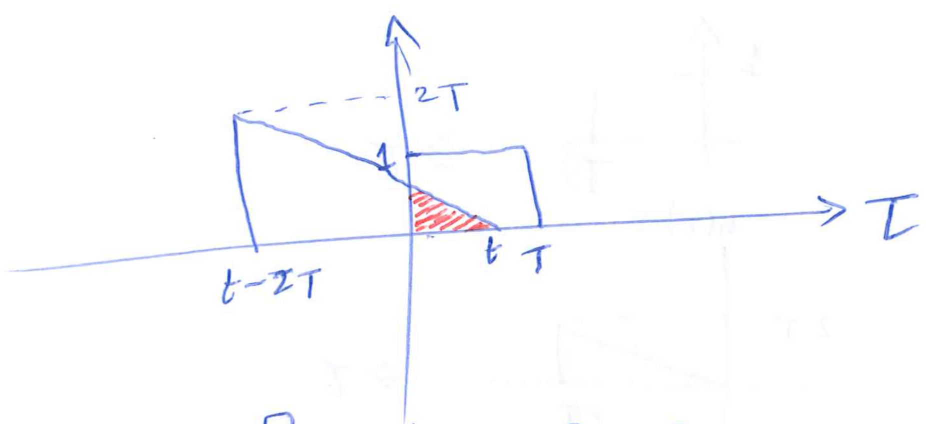
5-



$t<0$ there is no overlaping

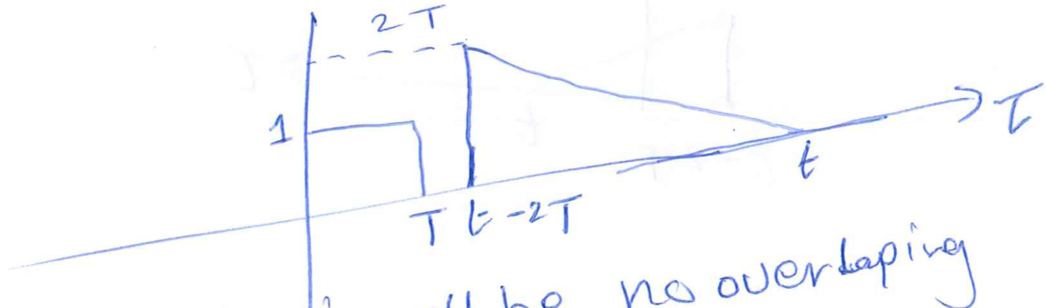
the integration = 0

$y(t) = 0$



$t > 0$ & $t < 3T$

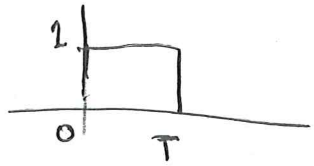
There will be over laping and the integration can be calculated



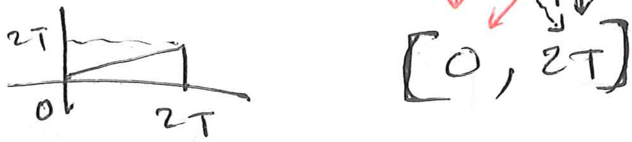
If $> 3T$ there will be no overlaping and the integration = 0

- let us determine the intervals

x(t)



h(t)

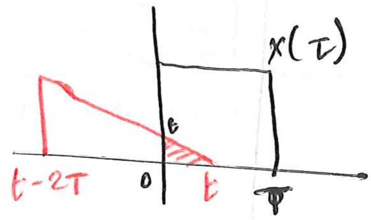


$\therefore$ y(t) has these intervals $[0, T, 2T, 3T]$

① $y(t) = 0$ for $t < 0$
② $y(t) = 0$ for $t > 3T$
} There is no overlaping

③ for $0 < t < T$

$y(t) = \text{Area}$
$= \frac{1}{2}(t)(t)$
$= \boxed{\frac{1}{2}t^2}$



or

Using the equation of the function

slope $m = \frac{0-t}{t-0} = \frac{-t}{t} = -1$

equation of the function is $y - 0 = m(x - t)$ $\quad x = \tau$

$y = -1(\tau - t)$
$= t - \tau$

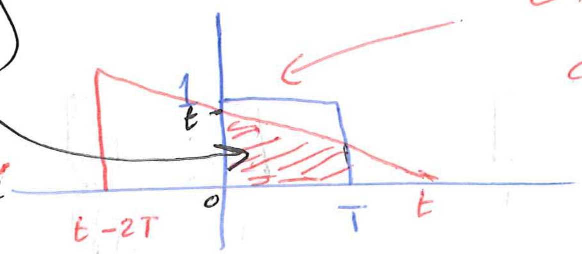
$$\therefore y(t) = \int_0^t (t - \tau)\, d\tau = t\tau - \frac{\tau^2}{2}\Big|_0^t = t^2 - \frac{t^2}{2} = \boxed{\frac{1}{2}t^2}$$

4) for $T < t < 2T$

$y(t) = \text{Area}(\text{overlaping})$

$= \frac{1}{2}t^2 - \left(\frac{1}{2}(T-t)^2\right)$

$= \cancel{\frac{1}{2}t^2} - \frac{1}{2}T^2 + Tt - \cancel{\frac{1}{2}t^2}$



انتبه جيداً أن convolution يساوي مساحة حاصل ضرب الاقترانين

$\therefore y(t) = Tt - \frac{1}{2}T^2$

or by integration

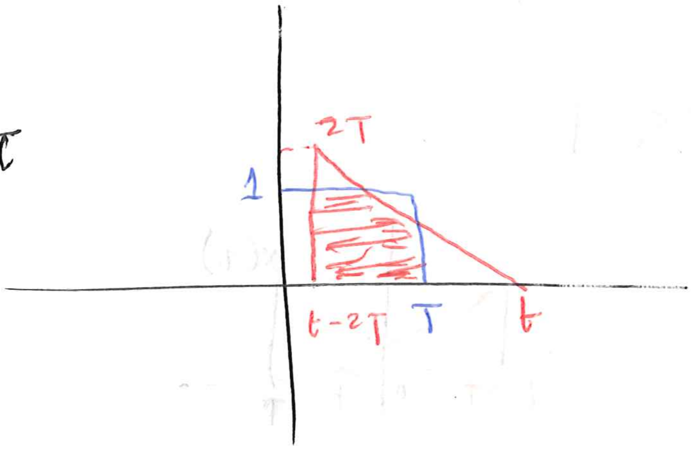
the second function here equals 1 but if the value is changed, It should be applied here integration.

$y(t) = \int_0^T (t - \tau)\, d\tau$

$= t\tau - \frac{\tau^2}{2} \Big|_0^T = tT - \frac{1}{2}T^2$

5) for $2T < t < 3T$

$y(t) = \int_{t-2T}^{T} (t - \tau)\, d\tau$



$= t\tau - \frac{\tau^2}{2} \Big|_{t-2T}^{T}$

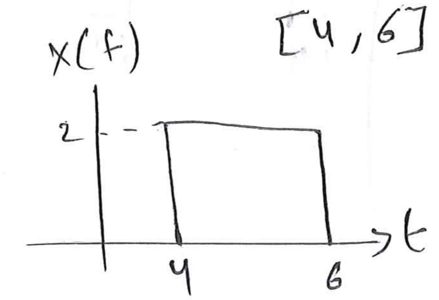
$= \left(tT - \frac{T^2}{2}\right) - \left[t(t-2T) - \frac{(t-2T)^2}{2}\right] = \cancel{-\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2}$

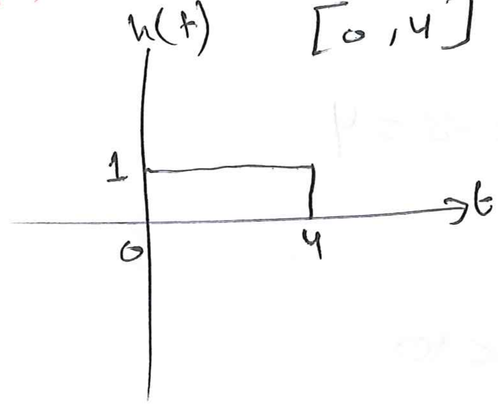
$= tT - \frac{T^2}{2} - \cancel{t^2} + 2tT + \cancel{t^2} - 4tT + 4T^2$

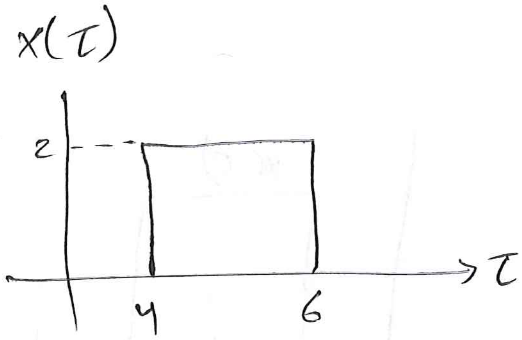
$= -tT + \frac{7}{2}T^2$

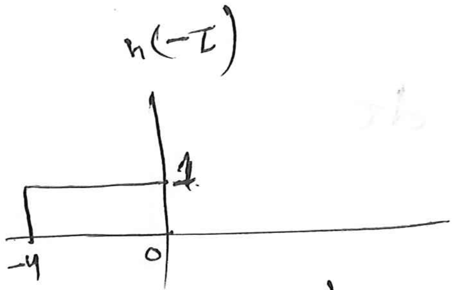
$y(t) = \begin{cases} \frac{1}{2}t^2 \\ tT - \frac{1}{2}T^2 \\ -tT + \frac{7}{2}T^2 \end{cases}$

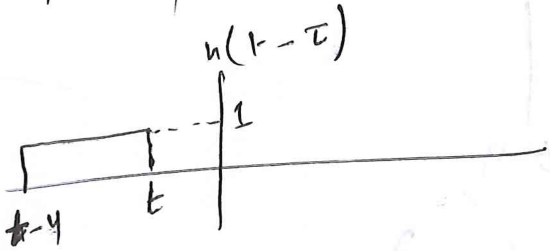
Example:- For the given signal $x(t) = 2\Pi\left(\frac{t-5}{2}\right)$, find the response $y(t)$, given that the impulse response $h(t) = \Pi\left(\frac{t-2}{4}\right)$









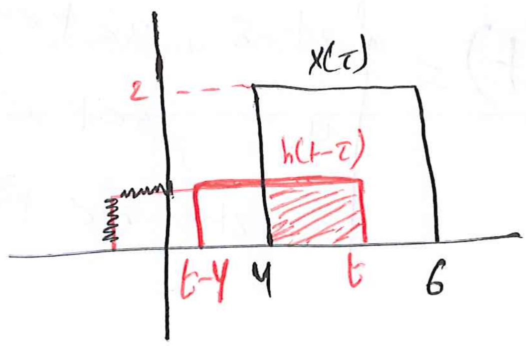


$y(t)$ has these intervals $[4, 6, 8, 10]$

① $y(t) = 0$ for $t < 0$

② $y(t) = 0$ for $t > 10$

③ for $0 < t < 6$



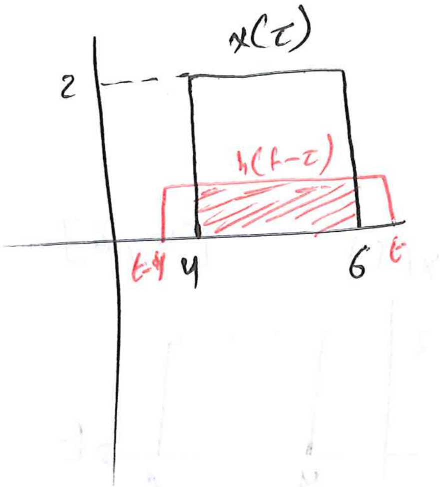
$$y(t) = \int_4^t (1)(2)\, d\tau$$

$$= 2\tau \Big|_4^t = 2(t-4) = 2t - 8$$

④ for $6 < t < 8$

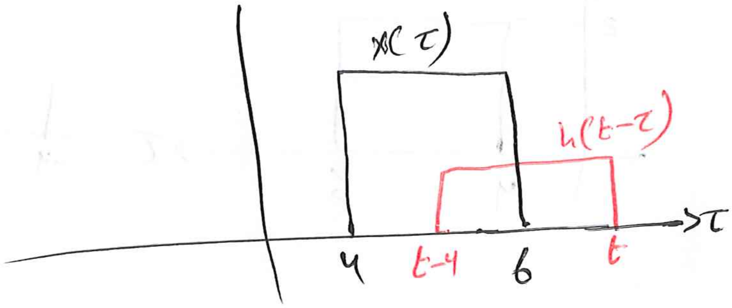
$$y(t) = \int_{4}^{6} (1)(2)\, d\tau$$

$$= 2\tau \Big|_{4}^{6} = 12 - 8 = 4$$



⑤ for $8 < t < 10$

$$y(t) = \int_{t-4}^{6} (1)(2)\, d\tau$$



$$= 2\tau \Big|_{t-4}^{6}$$

$$= 12 - 2(t-4)$$

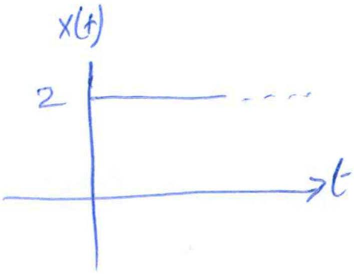
$$= 12 - 2t + 8$$

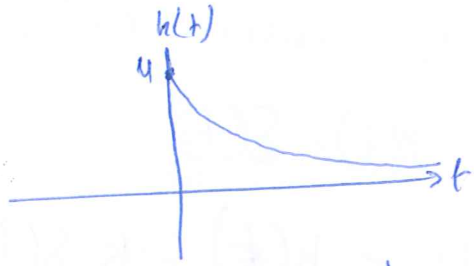
$$= 20 - 2t$$

$$y(t) = \begin{cases} 2t - 8, & 4 < t < 6 \\ 4, & 6 < t < 8 \\ 20 - 2t, & 8 < t < 10 \end{cases}$$

B Analytical Evaluation :-

Examples:- $x(t) = 2u(t)$ $\qquad$ $h(t) = 4e^{-3t}u(t)$





$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} 2u(\tau) \cdot 4\, e^{-3(t-\tau)}\, u(t-\tau)\, d\tau$$

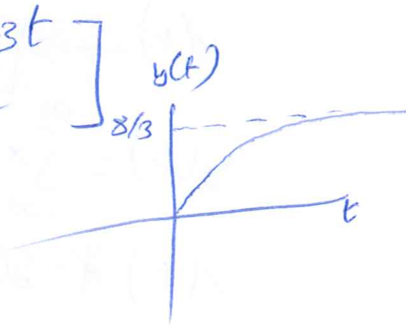
$$u(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases} \qquad u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases}$$

$\therefore$ period $0 < \tau < t$ $\quad y(t)$ has output
and zero otherwise

$$y(t) = \int_{-\infty}^{0} 8\, \cancel{e^{-3t}\, e^{3\tau}}^{\,0}\, d\tau + \int_{0}^{t} 8\, e^{-3t}\, e^{3\tau}\, d\tau + \int_{t}^{\infty} 8\, \cancel{e^{-3t}\, e^{3\tau}}^{\,0}\, d\tau$$

$$= \int_{0}^{t} 8e^{-3t} e^{3\tau}\, d\tau = 8e^{-3t} \int_{0}^{t} e^{3\tau}\, d\tau = 8e^{-3t}\, \frac{1}{3} e^{3\tau} \Big|_{0}^{t}$$

$$= \frac{8}{3} e^{-3t} \left[e^{3t} - 1 \right] = \frac{8}{3} \left[1 - e^{-3t} \right]$$



$$y(t) = \begin{cases} \frac{8}{3}\left[1 - e^{-3t}\right] & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Properties of Continuous Time LTI System

To show that $h(t)$ is descriping the system

[1] memory and memory less

$y(t)$ depends only on $x(t)$

if $x(t) = \delta(t)$

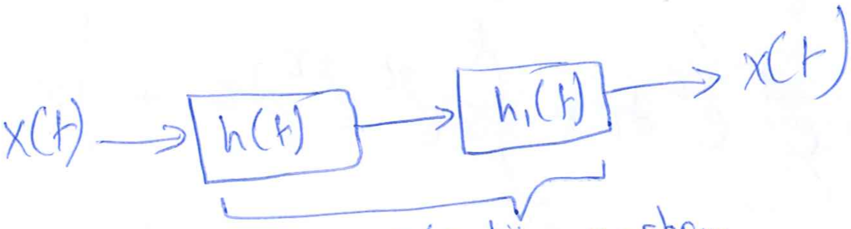
$y(t) = h(t) = K\,\delta(t)$ is a memory less system if and only if $K = \int_{-\infty}^{\infty} h(t)\,dt$ = Constant other wise

LTI system is a memory system

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

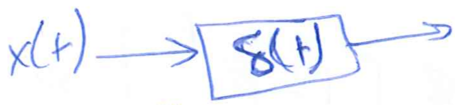
depends on other intervals thus memory system

[2] Invertible / Non-invertible



Identity system

the system will be invertable if $h(t) * h_1(t) = \delta(t)$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$$

$x(t) * \delta(t) = x(t)$ $x(t) * \delta(t-t_0) = x(t-t_0)$

The system (S) will be invertible if there exist a function $h_1(t)$ such that

$$h(t) * h_1(t) = \delta(t)$$

$h(t) = \delta(t - t_0)$

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow \delta(t-t_0) \qquad \underset{\delta(t+t_0)}{\boxed{h_1(t)}} \longrightarrow \delta(t)$$

$\boxed{3}$ Causal / non Causal

$$\delta(t) \longrightarrow \boxed{S} \longrightarrow h(t)$$

Can the output appear before $\delta(t)$ exist at $t=0$

If $h(t) = 0$ for $t<0$ then the system is Causal

$\boxed{4}$ stable / non stable

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau \qquad \text{if } |x(t-\tau)| < M$$

the system will be stable if $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$

Example :- Check the system stability if $h(t) = e^{-3t} u(t)$

To check the stability :-

$$\int_{-\infty}^{\infty} h(t)\, dt < \infty$$

$$\int_{-\infty}^{\infty} e^{-3t}\, dt\, u(t) = \int_{0}^{\infty} e^{-3t}\, dt = \frac{e^{-3t}}{-3} \Big|_{0}^{\infty} = \frac{1}{3} < \infty$$

$\therefore$ stable

If $h(t) = e^{3t} \Rightarrow$ the system is not stable

$h(t) = 10\,u(t)$ check the stability

$$\int_{-\infty}^{\infty} h(t)\, d\tau = \int_{0}^{\infty} 10\, dt = 10t \Big|_{0}^{\infty} = \infty \quad \text{unstable}$$

* Unit step response :-

$$\delta(t) \longrightarrow \boxed{LTI} \longrightarrow h(t) \qquad u(t) = \int_{-\infty}^{\infty} \delta(t)$$

$$u(t) \longrightarrow \boxed{LTI} \longrightarrow a(t) \qquad \therefore \; a(t) = \int_{-\infty}^{\infty} h(t)$$

$$a(t) = \int_{-\infty}^{\infty} u(t)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, \underline{u(t-\tau)}\, d\tau$$

defined from zero to (t)

$$= \int_{0}^{\infty} h(t-\tau)\, d\tau \qquad = \int_{0}^{t} h(\tau)\, d\tau$$

If $h(t)$ is Causal $\Rightarrow h(t) = 0$ for $t < 0$

viseversa if we take the derivatie of the step response we can calculate the impulse responsy

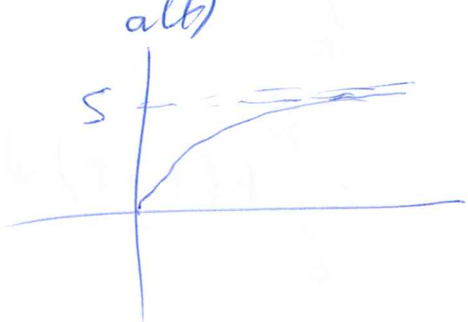
$$h(t) = \frac{d a(t)}{dt}$$

Examples: let $h(t) = 10e^{-2t}u(t)$

find $a(t)$ the step response

$$\therefore a(t) = \int_0^t h(\tau)\, d\tau$$

$$= \int_0^t 10e^{-2t} d\tau = \frac{-10}{2} e^{-2\tau} \Big|_0^t = 5[1-e^{-2t}]u(t)$$



* Ramp response

using the sam Concept

since $r(t) = \int u(t) \to$ ramp response $= \int a(t)$

* Frequency Response of LTI system

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$h(t)$

if the input $x(t)$ is complex

$$x(t) = e^{j\omega t}$$

then $y(t) = x(t) * h(t)$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}\, h(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \underbrace{e^{j\omega t}}_{\text{constant}}\, e^{-j\omega\tau}\, h(\tau)\, d\tau$$

$$= \underbrace{e^{j\omega t}}_{\text{same as the input}} \underbrace{\int_{-\infty}^{\infty} e^{-j\omega\tau}\, h(\tau)\, d\tau}_{\text{let us call this } H(\omega)}$$

frequency response

$$H(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau}\, h(\tau)\, d\tau$$

(Fourier Transform)

$$H(\omega) = F\{h(t)\}$$

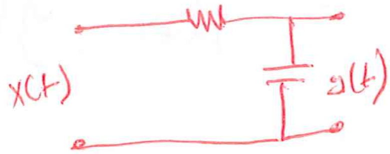
$$y(t) = x(t) \cdot H(\omega)$$

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega) = H(\omega) \cdot X(\omega)$$

$H(\omega)$ is a complex quantity

$$H(\omega) = \underset{\text{amplitude}}{|H(\omega)|}\, e^{j \underbrace{\theta_{H(\omega)}}_{\text{phase}}} = H(\omega) \angle \theta$$

Example : Find the frequency response of the given system $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$



$$H(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} h(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j\omega\tau} \left(\frac{1}{RC} e^{-\tau/RC} \right) u(\tau)\, d\tau$$

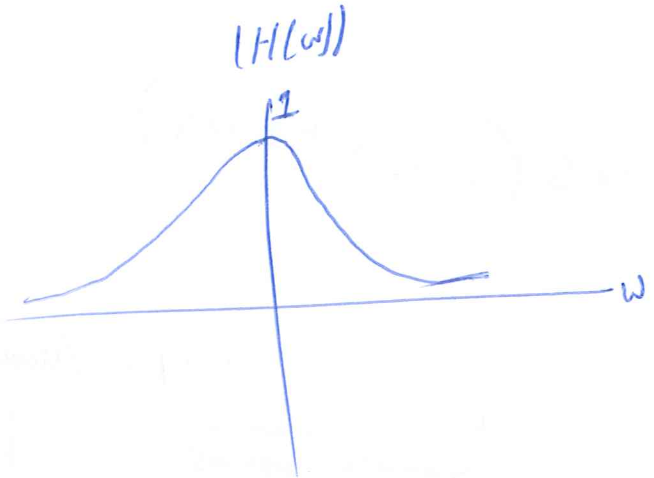
$$= \frac{1}{RC} \int_{0}^{\infty} e^{-\left(\frac{1}{RC} + j\omega\right)\tau} d\tau = \frac{1}{RC} \left[\frac{-1}{\frac{1}{RC} + j\omega} \right] e^{-\left(\frac{1}{RC}+j\omega\right)\tau} \Big|_0^{\infty}$$

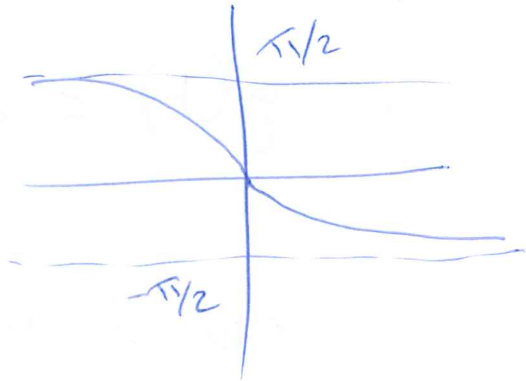
$$= \frac{1}{RC} \left[\frac{-1}{\frac{1}{RC} + j\omega} \right] (0 - 1)$$

$$= \frac{1}{RC} \left[\frac{1}{\frac{1}{RC} + j\omega} \right] = \frac{1}{1 + j\omega RC}$$

$$= \frac{1}{\sqrt{1 + (\omega RC)^2}\; e^{j \tan^{-1}(\omega RC)}} = \frac{1}{\sqrt{1 + (\omega RC)^2}}\, e^{-j \tan^{-1}(\omega RC)}$$

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}} \qquad \Theta_{H(\omega)} = -\tan^{-1}(\omega RC)$$





So for any LTI system

$$\underset{x(t)}{A\cos(\omega_0 t)} \longrightarrow \boxed{H(\omega)} \longrightarrow y(t)$$

$$y(t) = A\,|H(\omega_0)|\cos(\omega_0 t + \Theta_{H(\omega_0)})$$

Examples:- let $\frac{1}{2\pi RC} = 1KHz$ $\quad A = 1$

$$RC = \frac{1}{2\pi(1000)} = \frac{1}{2000\pi}$$

for $x(t) = \cos(\underbrace{2000\pi}_{\omega} t) \rightarrow$ find $y(t)$

$\cancel{y(t) = 1\ E}$ $\quad y(t) = A\cdot|H(\omega)|\cos(2000\pi t + \Theta)$

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}} = \frac{1}{\sqrt{1+\left(2000\pi\left(\frac{1}{2000\pi}\right)\right)^2}} = 0.707$$

$$\phi(2000\pi) = -\tan^{-1}\left(2000\pi \cdot \frac{1}{2000\pi}\right) = -45$$

$$H(\omega) = 0.707\, e^{-j45}$$

$$\therefore\ y(t) = 0.707 \cos(2000\pi t - 45)$$

Find the response of $x(t) = \underbrace{\cos(2000\pi t)}_{\text{same as previous}} + \cos(4000\pi t)$

$$|H(4000\pi)| = \frac{1}{\sqrt{1+\left(4000\pi \frac{1}{2000\pi}\right)}} = 0.45$$

$$\phi(4000\pi) = -\tan^{-1}\left(4000\pi \frac{1}{2000\pi}\right) = -63.43^{\circ}$$

$$\therefore\ y(t) = 0.707 \cos(2000\pi t - 45) + 0.45 \cos(4000\pi t - 63.43)$$

* System modeling & Simulation

Example :- plot the simulink model of the following differential equation :-

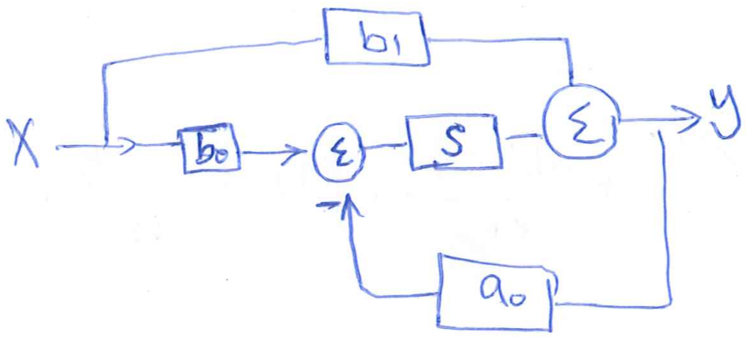
$$\frac{dy}{dt} + a_0 y = b_1 \frac{dx}{dt} + b_0 x$$

$$y' + a_0 y = b_1 x' + b_0 x$$

$$y' = b_1 x' + b_0 x - a_0 y$$

$$\int y' = \int b_1 x' + \underbrace{\int b_0 x - a_0 y}_{q_0}$$

$$y = b_1 x + q_0$$



Example :-

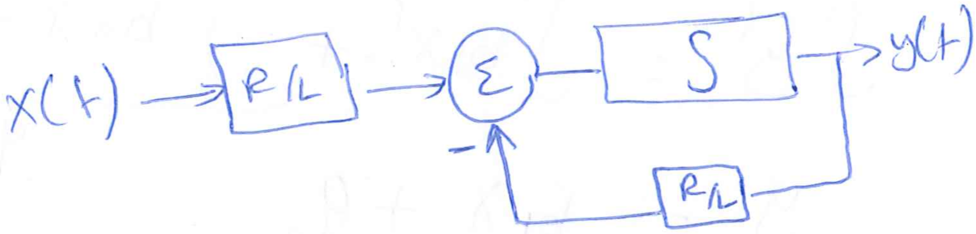
$$-x(t) + \frac{L}{R}\frac{dy(t)}{dt} + y(t) = 0$$

$$-x(t) + \frac{L}{R}\, y' + y(t) = 0$$

$$\frac{L}{R}\, y' = x(t) - y(t)$$

$$y' = \frac{R}{L}x(t) - \frac{R}{L}y(t)$$

$$y(t) = \int \left(\frac{R}{L}x(t) - \frac{R}{L}y(t) \right)$$



Example :-

$$2\frac{d^3y(t)}{dt^3} - 8\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 2y(t) = 4\frac{dx(t)}{dt} + 2x(t)$$

/2

$$\frac{d^3y(t)}{dt^3} - 4\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = 2\frac{dx(t)}{dt} + x(t)$$

$$y''' - 4y'' + 2y' + y = 2x' + x$$

$$y''' - 4y'' + 2y' - 2x' = x + y$$

$$\int y''' - 4\int y'' + 2\int y' - 2\int x' = \underbrace{\int (x+y)}_{q_0}$$

$$y'' - 4y' + 2y - 2x = q_0$$

$$y'' - 4y' = q_0 + 2x - 2y$$

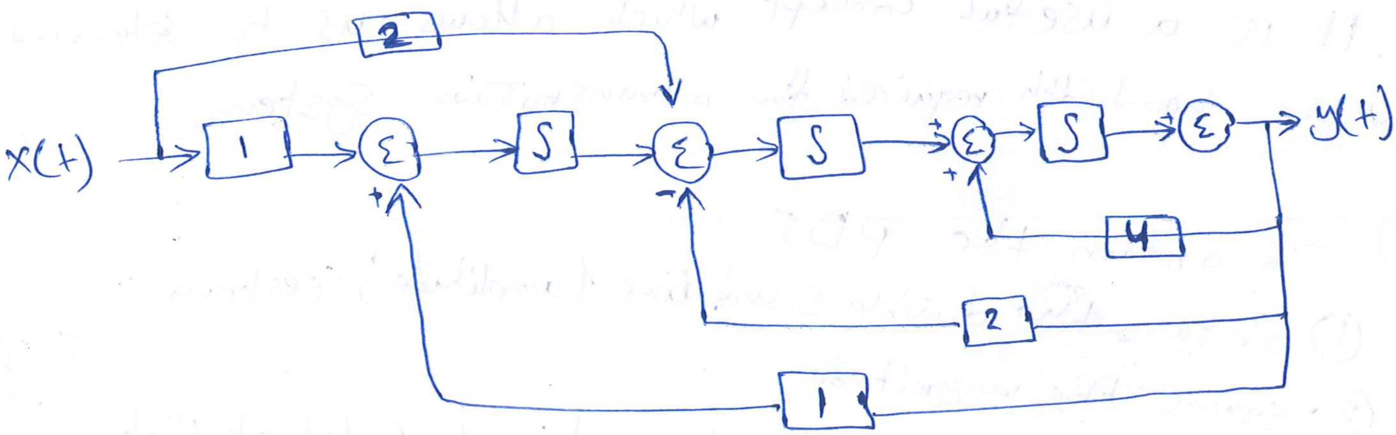
$$\int y'' - 4\int y' = \underbrace{\int (q_0 + 2x - 2y)}_{q_1}$$

$$y' - 4y = q_1$$

$$y' = q_1 + 4y$$

$$\int y' = \underbrace{\int (q_1 + 4y)}_{q_2}$$

$$y = q_2$$



* Energy and Power spectral Density

Energy spectral density :- Measures the distribution of signal energy E
(ESD)

over frequency.

$$E = \int_{-\infty}^{\infty} G(f)\, df$$

where E is the signal total energy.

It will be discussed in detailes in Chapter 4

Power spectral density :- Measures the distribution of power P
(PSD)

as a function of frequency

$$P = \int_{-\infty}^{\infty} S(f)\, df$$

where P is the average power of the signal.

It is a useful concept which allows us to determine the bandwidth required for a transmission system.

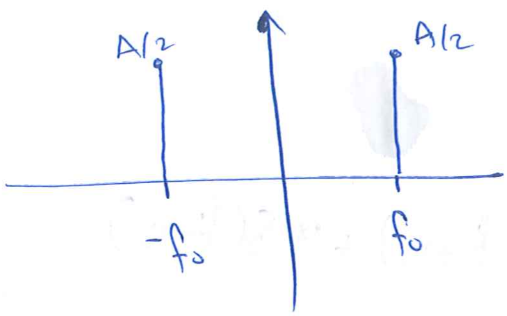
To obtain the PDS

1. obtain the double sided line (amplitude) spectrum
2. square the magnitude
3. Multiply the squared value by unit impulse located at that particular frequency.

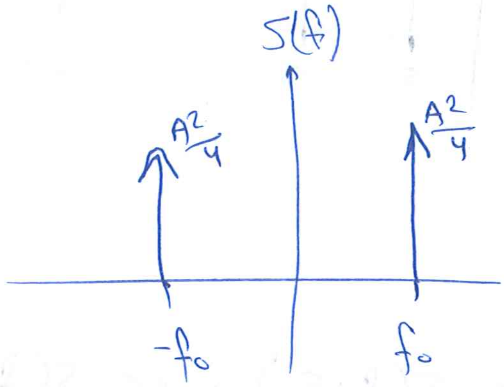
Example:

$$x(t) = A\cos(2\pi f_0 t + \theta)$$

$$= \frac{A}{2} e^{j\theta} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j\theta} e^{-j2\pi f_0 t}$$



$\Rightarrow$



$$S(f) = \frac{A^2}{4}\delta(f+f_0) + \frac{A^2}{4}\delta(f-f_0)$$

$$P = \int_{-\infty}^{\infty} S(f)\, df$$

$$= \int \frac{A^2}{4}\delta(f+f_0) + \int \frac{A^2}{4}\delta(f-f_0)\, df$$

$$= \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2} \quad W$$

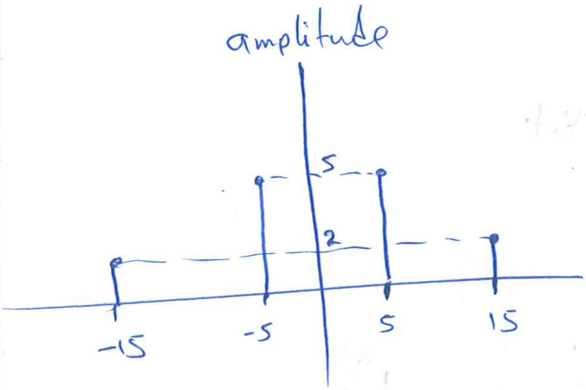
Example:

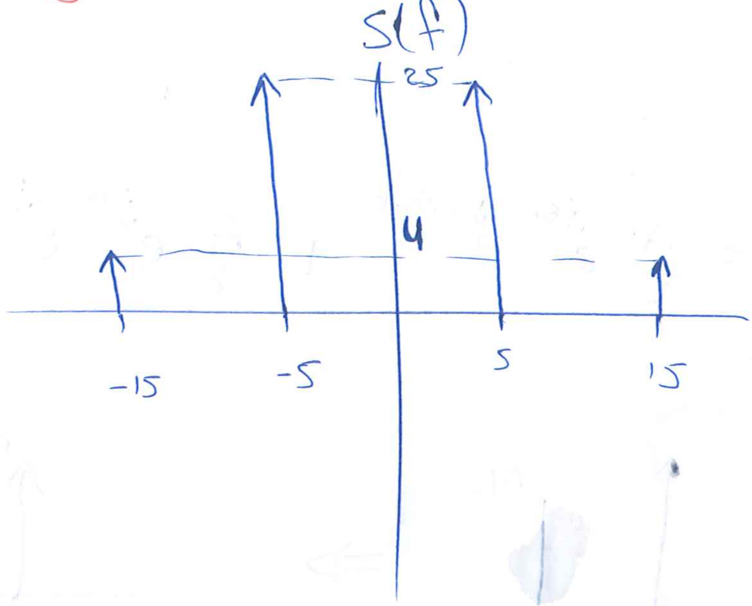
$$x(t) = 10\cos(10\pi t + \pi/7) + 4\sin(30\pi t + \pi/8)$$

(1) plot its power spectral density

(2) Compute the power lying within a frequency band from 10 Hz to 20 Hz

① the power spectral density





$$S(f) = 25\,\delta(f+5) + 25\,\delta(f-5) + 4\,\delta(f+15) + 4\,\delta(f-15)$$

$$P = \int_{-\infty}^{\infty} S(f)\,df = 25 + 25 + 4 + 4 = 58\ W$$

② the power lying within a frequency band from 10Hz to 20Hz

is $= 4 + 4 = 8\ W$