Control systems

System Transient Performance

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Linear Time invariant system in Laplace and time domains:

Poles and Zeros of LTI Systems:
Given the transfer function of a proper system i(primitive rational function):

$$T(s) = \frac{\beta_m s^m + \beta_{m-1} s^{m-1} + \cdots + \beta_1 s^1 + \beta_0}{\alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s^1 + \alpha_0} \quad \text{with } m < n$$

System Zeros:

A system zero is defined as the value s_z at which $|T(s_z)| = 0$.
A system zero can be a zero at finite or infinite.
A proper system has $n - m$ zeros at infinite, that is those that satisfy the relation $\lim_{s \to \infty} |T(s)| = 0$.

System Poles:

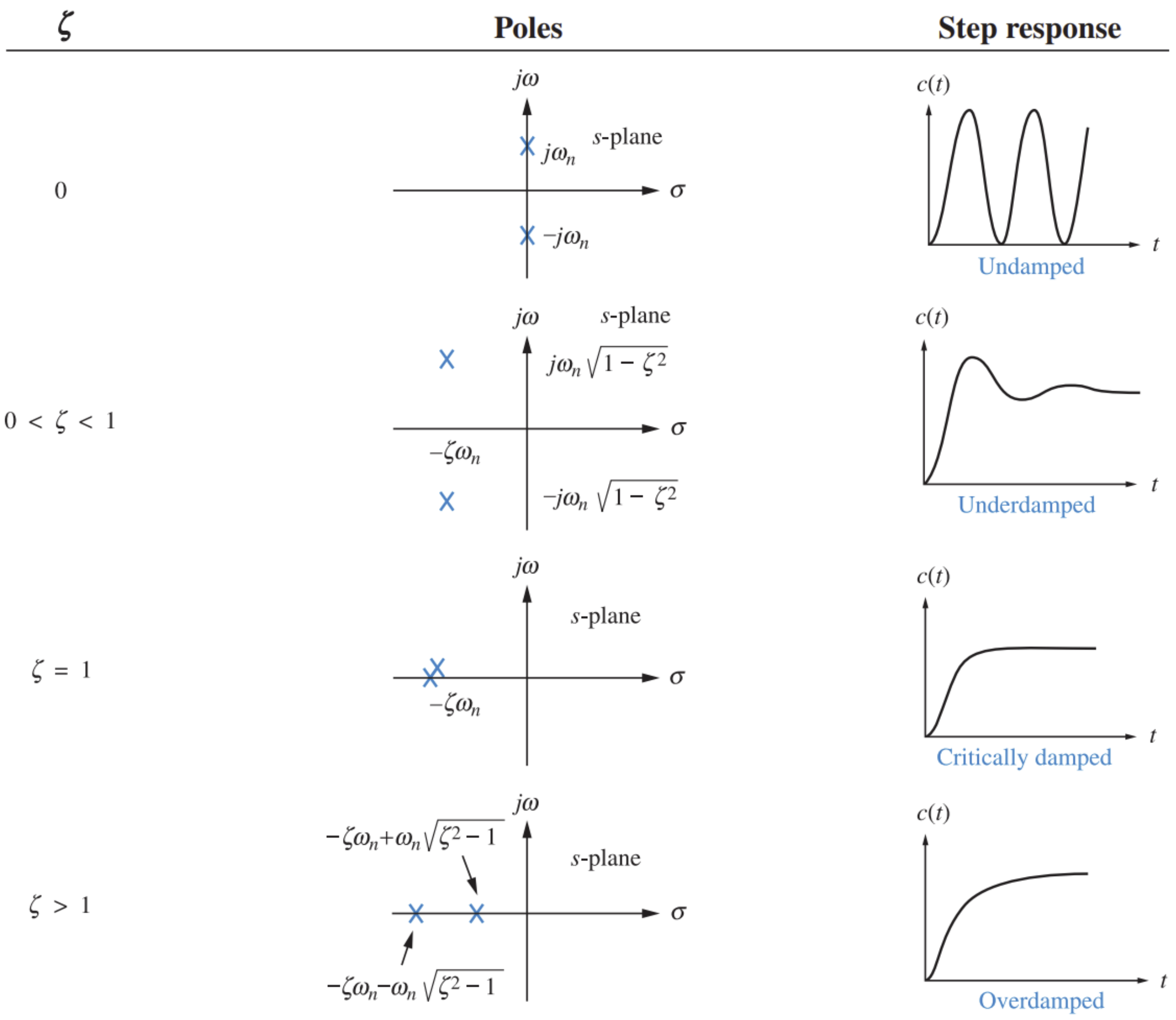
A system pole is defined as the value s_p at which $\lim_{s \to s_p} |T(s)| = \infty$.

A system pole can be a pole at finite or infinite.
An improper system (improper: $m \geq n$) has $m - n$ poles at infinite, that is those that satisfy the relation $\lim_{s \to \infty} |T(s)| = \infty$

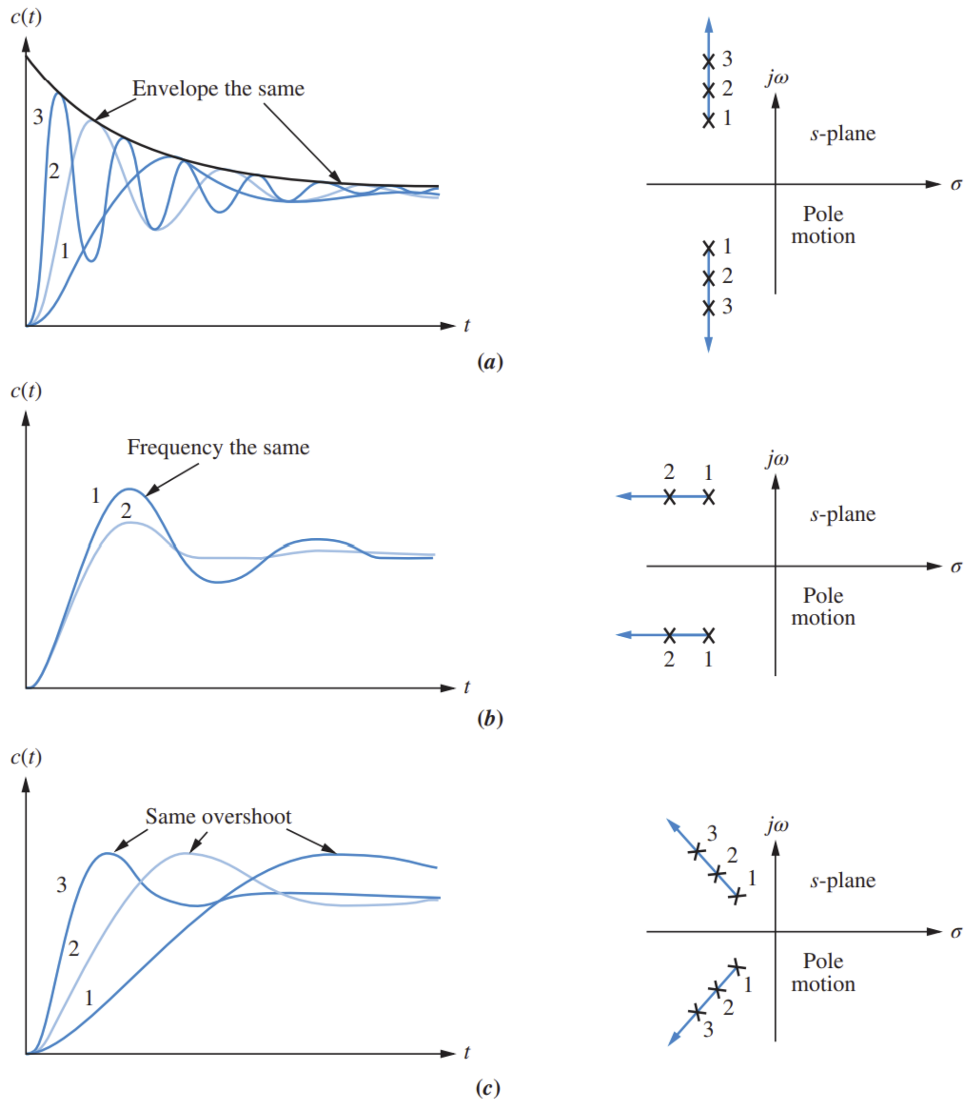
Effect of poles on system response:

The poles number and locations (system transfer function roots) determine the shape and the time performance of the transient response:

- Left-side poles generate a response that vanishes for $t \to \infty$, whereas right-side poles transient diverges
- Real-axe poles do not produce oscillation in the time response.
- Imaginary axe poles produce an undamped oscillation response.
- Complex poles produce oscillation in the response.

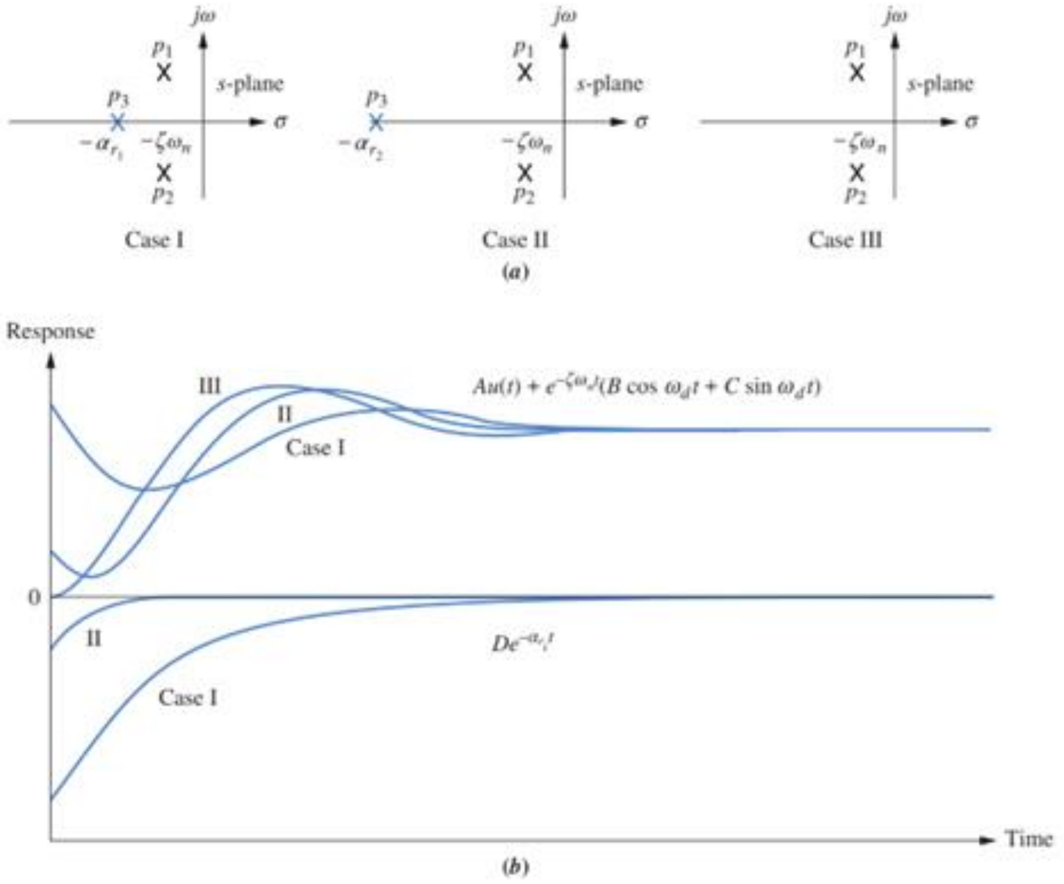


- Transient time performance depends on the relative distance between the imaginary axe and the pole location. That is the magnitude of the real part of the pole. Higher distance → Higher performance and faster transient. The time constant of a pole s_p is defined as $\tau_p = -\frac{1}{Re(s_p)}$
- Oscillation frequency depends on the relative distance between the real axe and the pole location. That is the magnitude of the imaginary part of the pole. Higher distance → Higher oscillation frequency and smaller period with higher density of oscillation cycles.
- Poles that are located on the same line have different time performance and oscillation frequency but equal damping ratio and relative overshoot value.



Dominant Poles:

- The set of dominant poles are those proximal to the imaginary axe and from the more distal poles with $\min(\tau_{dom}) > 5 \times \max(\tau_{non-dom})$. The dominant poles has slower transient response and thus they are the objective of the control problem.
- Considering the dominant poles reduces the order of the control system and the design of the controllers.



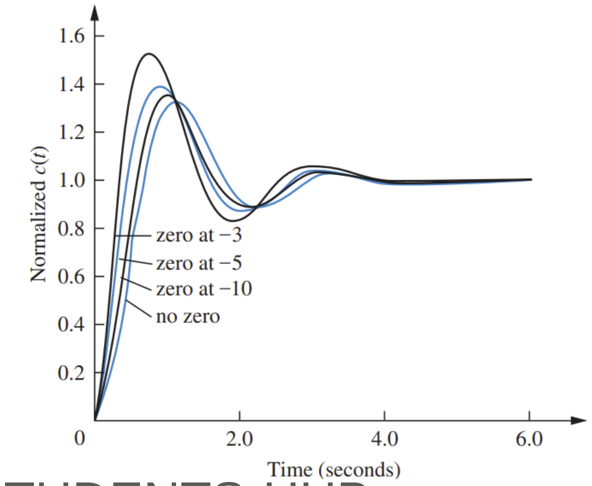
Component responses of a three-pole system: **a.** pole plot; **b.** component responses: Nondominant pole is near dominant second-order pair (Case I), far from the pair (Case II), and at infinity (Case III)

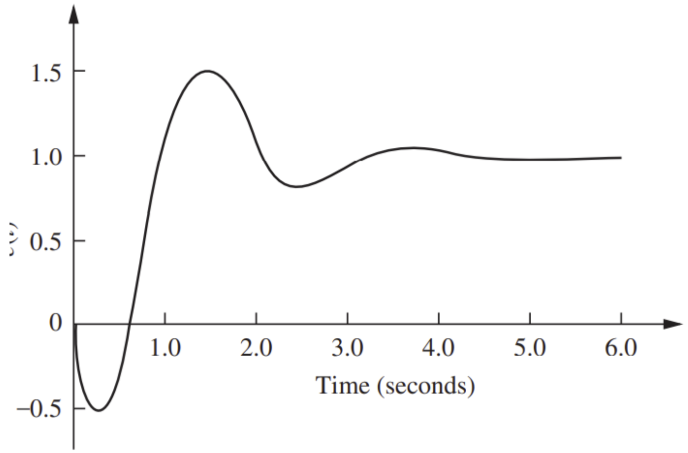
Pole-Zero Cancellation: zeros and poles can be set at the same position to cancel the effects of each other. Cancellation can be employed in controller design to cancel undesired effects or to reduce the order of the system(if it is a design degree of freedom)

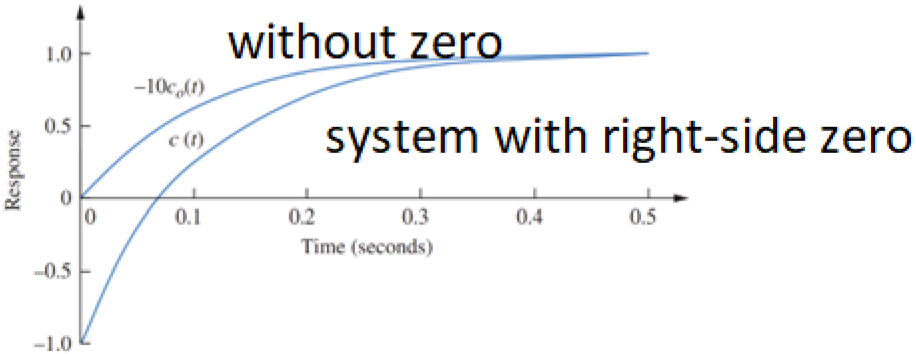
Effect of Zeros:

$$(s + a)C(s) = sC(s) + aC(s)$$

- The zeros affect the response amplitude.
- The effects of the zeros are more evident when they are more proximal to the dominant poles (zeros with smaller real part has a higher time constant and has a more evident effect on the system response).
- The zeros affect the response phase.
- A real zero (or the real part of a complex zero) introduces a derivative and proportional effect in the response without zero.
- For more distal zeros (from the imaginary axe) the proportional effect is higher than the derivative effect (fast zero effect). For the nearer zeros, the derivative effect is higher than the proportional one.
- Slower zeros cause higher signal overshoot because of the added positive value of the derivative.
- A left-side complex zero has a positive phase and thus an anticipation effect.
- A right-side complex zero has a negative phase and thus introduces a delay effect.
- A right-side zero with a smaller derivative effect than the proportional part may cause initial phase inversion.
- Asymptotically stable systems with only left-side zeros are said to be minimum-phase systems



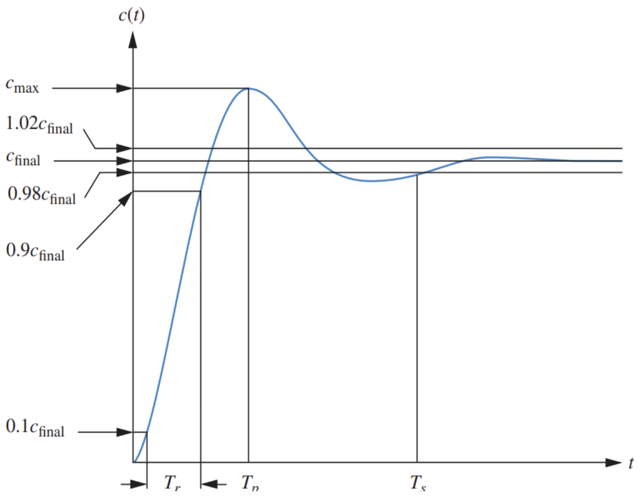




Performance parameters:

Performance parameters are used to set, evaluate, and compare the behavior of stable dynamic systems.

Time performance parameters:



- Rising time t_{ris}: the time necessary for the response to rise from 10% to 90% its final value.
- Delay-time t_d: the time necessary for the response to reach 50% of its final v
- Steady-state time (Settling time) t_{set}: at p% error: the time necessary for the response to reach and stay in $\pm 0.0p$ around its final value.
- Peak time t_{pn}: the time of the local maximum and minimum values of the response.
- Overshoot time t_{ov}: the time of the maximum deviation of the response from its final value.

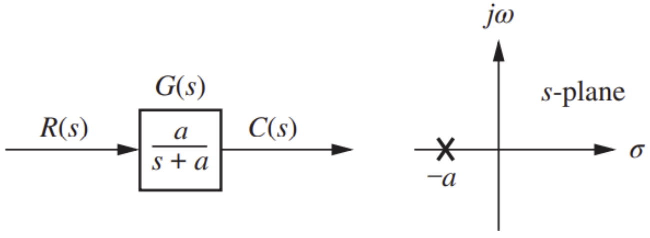
Value Performance parameters:

- Overshoot (OV): the maximum deviation between the response and its final steady state value. $OV(t_{ov}) = y_{max}(t_{ov}) - y_{final}$. This parameter depends on the input value.

- Relative Overshoot (OV_r): the ratio of the overshoot and the response final value. That is $OV_r = \frac{y_{max}(t_{ov}) - y_{final}}{y_{final}}$ independent of the input value but requires the knowledge of the final value.

- Percentage Overshoot ($OV_r\%$): the ratio of the overshoot and the response final value. That is $OV_r = \frac{y_{max}(t_{ov}) - y_{final}}{y_{final}} \times 100\%$

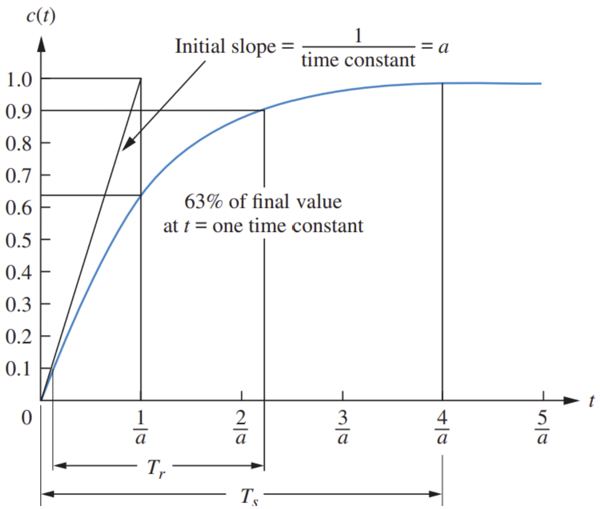
First Order System Step response:



(a) (b)

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

$$\boxed{c(t) = c_f(t) + c_n(t) = 1 - e^{-at}}$$



Time constant and steady-state approximation:

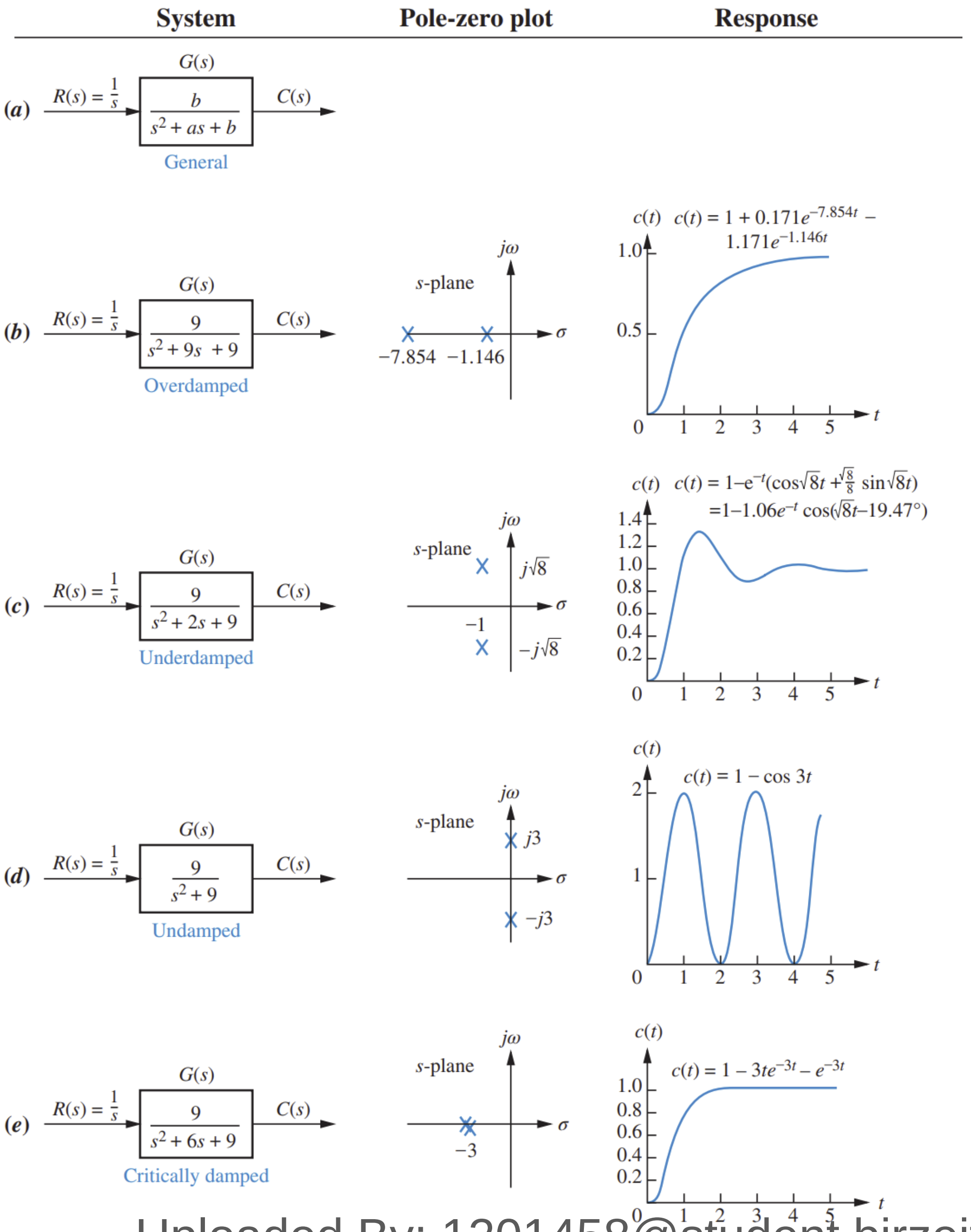
- The time constant is defined as $\tau = \frac{1}{a}$
- Steady-state time: is the time at which the steady state response is assumed to be reached accepting and tolerating a defined maximum error value (because operations with the system can not be done for $t \to \infty$).
- The most used in Engineering is $t_{steady=}4\tau$ with approximately $error_{steady=}2\%$

Second Order System:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The response of a stable second-order system has four shapes according to the classification of the poles of the system (roots of the characteristic algebraic equation.

- Overdamped oscillation response (no oscillation): real and different poles $\zeta > 1$ (positive discriminant)
- Critically damped response (change in convexity-start of oscillation: real and equal poles $\zeta = 1$ (discriminant=0)
- Undamped oscillation response (sustained oscillation): imaginary poles $\zeta = 0$ (negative discriminant with Re(pole)=0)
- Underdamped oscillation response:(damped oscillation): complex roots $0 < \zeta < 1$ (negative discriminant with $Re(pole) \neq 0)$



Second-order system-performance parameters

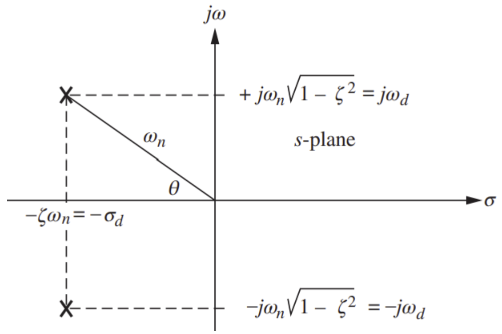
The underdamped response will be taken to determine the performance parameters because it has the maximum number of performance parameters.

Polar and cartesian representation:

a complex pair of system poles can be represented in:

cartesian form: $s_{1,2} = \alpha \pm j\omega_d$ α: *attenuation factor*, ω_d: *damped oscillation frequency*

Polar form: $s = \omega_n e^{j\theta}$ $\zeta = \cos\theta$: *attenuation ratio*, ω_n: *natural oscillation frequency*



Relations between polar and cartesian representations:

$$\alpha = \omega_n \cos\theta = \omega_n\zeta\,, \qquad \omega_d = \omega_n \sin\theta = \omega_n\sqrt{1-\zeta^2}$$

$$\omega_n = \sqrt{\alpha^2 + {\omega_d}^2}, \qquad \theta = -tan^{-1}\left(\frac{\omega_d}{\alpha}\right)$$

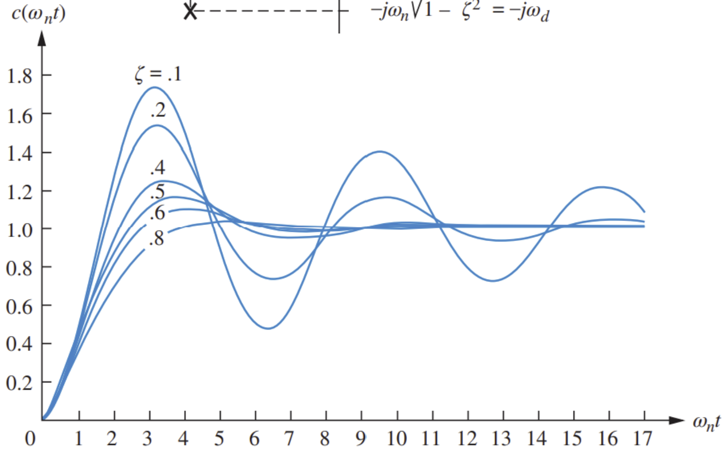
Under damped Step response:

$$C(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2\zeta\omega_n s+\omega_n^2}$$

By computing the parameters using partial fractions and adjusting the function form to have the Laplace cosine and sine expression:

$$C(s) = \frac{1}{s} - \frac{(s+\zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

using the Laplace inverse we obtain:



$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right)$$
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$$

$$\phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2}).$$

Performance parameters of the second-order system:

The most used performance parameters of the second-order system are the settling time, the overshoot time, and the overshoot value with all its variants.

Settling time at p% error:

To simplify computation it is assumed that the settling time is reaches at the first peak after the 0.0p, that is we consider a smaller error that satisfies the requirements. Thus, considering $c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$

we have to solve the equation: $\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_{set}} = 0.0p$

Solving for t_{set} we obtain: $t_{set} = \frac{-\ln(0.op\sqrt{1-\zeta^2})}{\zeta\omega_n}$ at 2% this result is approximated as $t_{set\_2\%} = \frac{4}{|\zeta\omega_n|} = \frac{4}{|\alpha|} = 4\tau$

Peak and overshoot time:

The peak time is periodic and obtained by computing equating the derivative of the step response c(t) to zero.

Since the underdamped response of the second order system is strictly decreasing, the overshoot time is obtained at the first peak value. Considering: $C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$ and the inverse of the derivative $\mathscr{L}[\dot{c}(t)] = sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$$\mathscr{L}[\dot{c}(t)] = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

Applying Laplace inverse $\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$

Setting the derivative equal to zero yields

$$\omega_n\sqrt{1-\zeta^2}\, t = n\pi \quad \leftrightarrow \quad t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Thus the overshoot time is: $\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} = \frac{\pi}{\omega_d}$

Overshoot evaluation: using $c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n \sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_n \sqrt{1-\zeta^2}\,t\right)$ and $\%OS = \frac{c_{\max} - c_{\text{final}}}{c_{\text{final}}} \times 100$

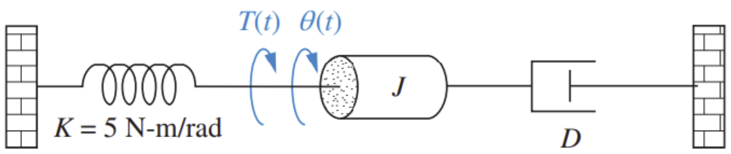
compute $c_{\max} = c(T_p) = 1 - e^{-(\zeta\pi/\sqrt{1-\zeta^2})}\left(\cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi\right) = 1 + e^{-(\zeta\pi/\sqrt{1-\zeta^2})}$

Applying $c_{final} = 1$ in the percentage overshoot equation we obtain $\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$

Moreover, in the design problem, we can compute the damping ratio necessary to obtain a specific percentage overshoot by:

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Example: Consider the following system and determine the moment of inertia and the damping coefficient to 20% overshoot and a 2% ERROR settling time of 2 seconds for a step torque input.



$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}} \rightarrow \quad \omega_n = \sqrt{\frac{K}{J}} \qquad 2\zeta\omega_n = \frac{D}{J}$$

$$T_s = 2 = \frac{4}{\zeta\omega_n} \rightarrow \zeta\omega_n = 2.$$

$$\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}} = 0.456 \rightarrow \frac{J}{K} = 0.052$$

$$\frac{J}{K} = 0.052$$

$$\frac{D}{J} = 4$$

$K = 5$ N-m/rad

$D = 1.04$ N-m-s/rad, and $J = 0.26$ kg-m^2