Natural logarithms

Exp sketch

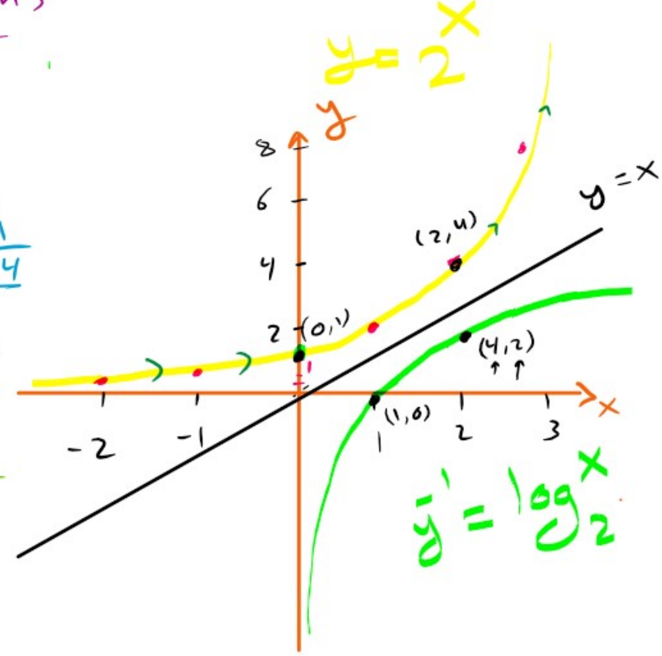
$y = 2^x \Rightarrow$

x	2^x
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

increasing on $\mathbb{R}$

concave up on $\mathbb{R}$

Exp sketch the inverse of 2^x



$\log_2 4 = 2$

$2^2 = 4$

$$y = \log_a u(x) = \frac{\ln u(x)}{\ln a}$$

$u(x) > 0$

$a > 0$

↑ base الأساس

$$y = \log_e u(x) = \frac{\ln u(x)}{\ln e} = \ln u(x) \qquad \ln e = 1$$

$$y = \log_e x = \frac{\ln x}{\ln e} = \frac{\ln x}{1} = \ln x$$

Def $y = \ln x = \int_1^x \frac{dt}{t} \Rightarrow y' = \frac{1}{x}$

e^x

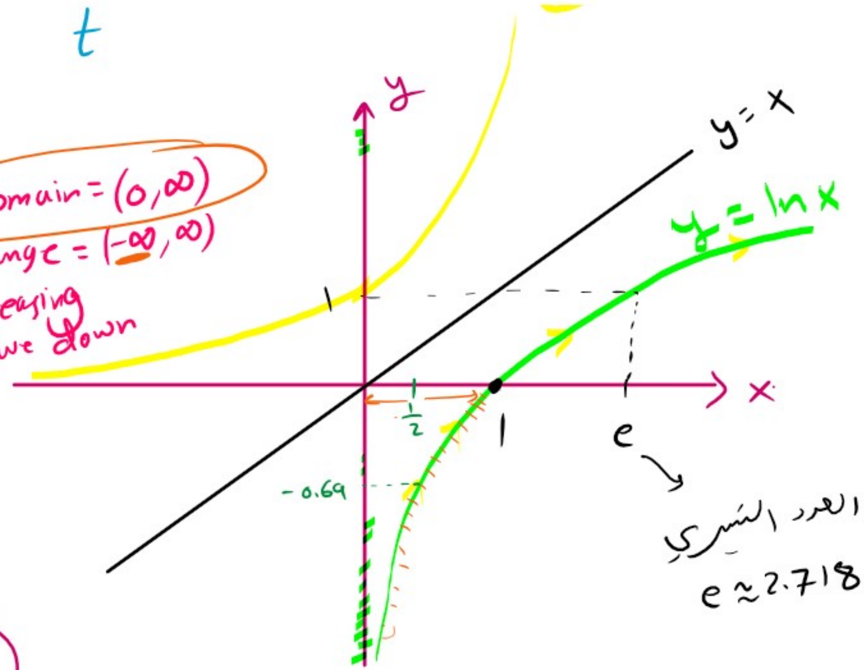
x	$\ln x$
0.5	-0.69
1	0
2	0.69
e	1
4	1.39
10	2.3

Domain $= (0, \infty)$

Range $= (-\infty, \infty)$

increasing

concave down



$\ln x \rightarrow x > 0$ domain

when $0 < x < 1 \Rightarrow \ln x < 0$

when $1 < x \Rightarrow \ln x > 0$

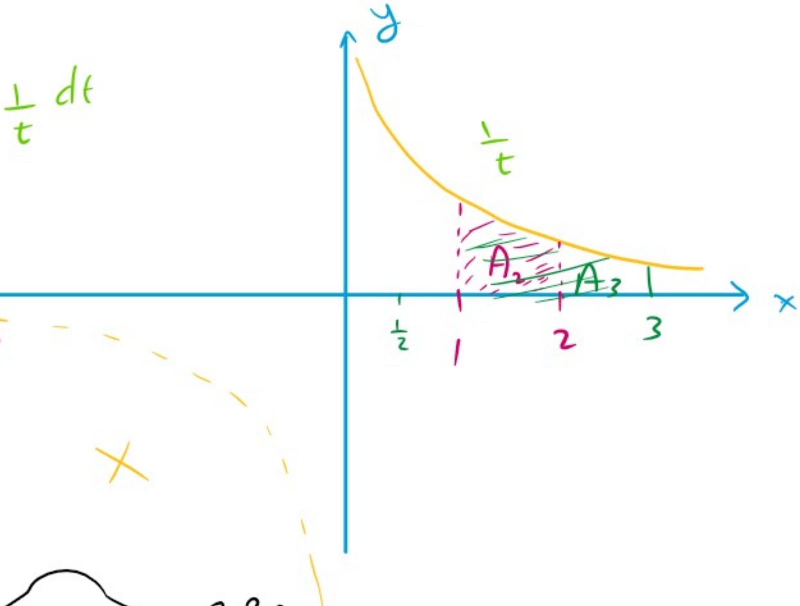
$x = 1 \Rightarrow \ln x = 0$

If $y = \ln x \Rightarrow y' = \dfrac{1}{x}$

$x > 0$

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt \approx 0.69 > 0$$



$$\ln 3 = \int_1^3 \frac{1}{t}\,dt \approx$$

$$\ln \tfrac{1}{2} = \int_1^{\frac{1}{2}} \frac{1}{t}\,dt = -\int_{\frac{1}{2}}^{1} \frac{dt}{t}$$ موجبة

$$\ln\frac{1}{2} = \ln 1 - \ln 2$$
$$= 0 - \ln 2 = -\ln 2 = -0.69$$

Properties of $\ln x = \log_e^x = \dfrac{\ln x}{\ln e} \to 1$

$a>0$, $b>0$

① $\ln ab = \ln a + \ln b$ Product Rule

② $\ln\frac{a}{b} = \ln a - \ln b$ Quotient Rule

③ $\ln\frac{1}{b} = \ln 1 - \ln b$ Reciproal Rule
$= 0 - \ln b$
$= -\ln b$

④ $\ln b^r = r \ln b$ Power Rule

① $\log_a xy = \log_a x + \log_a y$

② $\log_a \frac{x}{y} = \log_a^x - \log_a y$

③ $\log_a y^r = r \log_a y$

$\cup a$

inverse of 2^x is $\log_2 x$

inverse of $\log_3 x$ is 3^x

$\ln e = 1$

inverse of $\log_e x$ is e^x

$\ln x$

$\log_e x$

inverse of e^x is $\ln x$

inverse $\sqrt{5}^x$ is $\log_{\sqrt{5}} x$

Exf write $\ln\sqrt{13.5}$ in terms $\ln 2$ and $\ln 3$

$$\ln\sqrt{13.5} = \ln\sqrt{\frac{27}{2}} = \ln\frac{\sqrt{27}}{\sqrt{2}} = \ln\sqrt{27} - \ln\sqrt{2}$$

$$= \ln 27^{\frac{1}{2}} - \ln 2^{\frac{1}{2}}$$

$$= \frac{1}{2}\ln 27 - \frac{1}{2}\ln 2$$

$$= \frac{1}{2}\ln 3^3 - \frac{1}{2}\ln 2$$

$= \frac{1}{2}\ln 3^3 - \frac{1}{2}\ln 2$

$= \frac{3}{2}\ln 3 - \frac{1}{2}\ln 2$

Exp Find $\int_1^2 \frac{dx}{x} = \ln x \Big|_1^2$

$= \ln 2 - \ln 1$

$= \ln 2 - 0$

$= \ln 2$

$y = \ln u(x)$, $\underline{u(x) > 0}$

$u(x)$ diff

$y' = \frac{u'(x)}{u(x)}$ $\Rightarrow \int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + c$

Exp 1) $y = \ln x \Rightarrow y' = \frac{1}{x}$

2) $y = \ln\sqrt{x} \Rightarrow$ Find $y'(4)$

$\frac{1}{2}$ ... $= \frac{1}{2}\ln x$

$$y = \ln x^{\frac{1}{2}} = \frac{1}{2}\ln x$$

$$y' = \frac{1}{2}\,\frac{1}{x} \Rightarrow y'(4) = \frac{1}{2}\,\frac{1}{4} = \frac{1}{8}$$

or

$$y' = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}} = \frac{1}{2\sqrt{x}} \cdot \frac{1}{\sqrt{x}} = \frac{1}{2x}$$

$$y'(4) = \frac{1}{2(4)} = \frac{1}{8}$$

$$\int \frac{dx}{x} = \ln|x| + C$$

Exp

① $$\int_{-3}^{-2} \frac{dx}{x} = \ln|x| \Big|_{-3}^{-2} = \ln|-2| - \ln|-3|$$

$$= \ln 2 - \ln 3$$

$$= \ln \frac{2}{3}$$

46

② $$\int_{2}^{16} \frac{dx}{2x\sqrt{\ln x}}$$

$$u = \ln x$$

$$du = \frac{1}{x}\,dx$$

2

$du = \frac{1}{x}\,dx$

$x = 2 \Rightarrow u = \ln 2$

$x = 16 \Rightarrow u = \ln 16$

$$\int_{\ln 2}^{\ln 16} \frac{du}{2\sqrt{u}}$$

$$\sqrt{u}\,\Big|_{\ln 2}^{\ln 16} = \sqrt{\ln 16} - \sqrt{\ln 2}$$

$$= \sqrt{\ln 2^4} - \sqrt{\ln 2}$$

$$= \sqrt{4 \ln 2} - \sqrt{\ln 2}$$

$$= \sqrt{4}\,\sqrt{\ln 2} - \sqrt{\ln 2}$$

$$= 2\sqrt{\ln 2} - \sqrt{\ln 2}$$

$$= \sqrt{\ln 2}$$

(23)

Exp Find y' if $y = \ln(\ln x)$

Find $y'(e)$

$$y' = \frac{\frac{1}{x}}{\ln x}$$

Find $y(e)$

$$= \frac{1}{\ln e^{e}}$$

$$= \frac{1}{e \ln e}$$

$$= \frac{1}{e(1)}$$

$$= \frac{1}{e} \checkmark$$

$$\ln x$$

$$= \frac{1}{x} \cdot \frac{1}{\ln x} = \frac{1}{x \ln x}$$

$$= \frac{1}{\ln x^{x}}$$

Exp Use logarithmic diff to find $\frac{dy}{dx}$ if

(1) $y = \sqrt{x(x+1)}$

$$\frac{\sqrt{x}}{\text{الأول}} \quad \frac{\sqrt{x+1}}{\text{مشتقة}} +$$

$$\ln y = \ln \sqrt{x(x+1)}$$

$$= \ln \sqrt{x}\sqrt{x+1}$$

$$= \ln \sqrt{x} + \ln \sqrt{x+1}$$

$$= \ln x^{\frac{1}{2}} + \ln (x+1)^{\frac{1}{2}}$$

$$= \ln x^2 + \ln(x+1)$$

$$\ln y = \frac{1}{2}\ln x + \frac{1}{2}\ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2}\frac{1}{x} + \frac{1}{2}\frac{1}{x+1}$$

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right)$$

$$y' = \frac{y}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right)$$

$$= \frac{\sqrt{x(x+1)}}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right)$$

---

2) $y = t(t+1)(t+2)(t+3)$

$$\ln y = \ln t(t+1)(t+2)(t+3)$$

$$= \ln t + \ln(t+1) + \ln(t+2) + \ln(t+3)$$

$$\frac{y'}{y} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} + \frac{1}{t+3}$$

$$y' = y\left[\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} + \frac{1}{t+3}\right]$$

$$= t(t+1)(t+2)(t+3)\left[\frac{1}{t} + \cdots + \frac{1}{t+3}\right]$$

64

3

$$y = \frac{\theta \sin\theta}{\sqrt{\sec\theta}}$$

$$\ln y = \ln \frac{\theta \sin\theta}{\sqrt{\sec\theta}}$$

$$= \ln \theta \sin\theta - \ln\sqrt{\sec\theta}$$

$$= \ln\theta + \ln\sin\theta - \frac{1}{2}\ln\sec\theta$$

$$\frac{y'}{y} = \frac{1}{\theta} + \frac{\cos\theta}{\sin\theta} - \frac{1}{2}\frac{\cancel{\sec\theta}\tan\theta}{\cancel{\sec\theta}}$$

$$y' = y\left[\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right]$$

$$y' = y\left[\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right]$$

$$= \frac{\theta \sin\theta}{\sqrt{\sec\theta}}\left[\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right]$$

Recall

$$\int \sec^2 x \, dx = \boxed{\tan x} + C$$

$$\int \csc^2 x \, dx = \boxed{-\cot x} + C$$

$$\int \sec x \tan x \, dx = \boxed{\sec x} + C$$

$$\int \csc x \cot x \, dx = \boxed{-\csc x} + C$$

Q: Show that

1. $\int \tan x \, dx = -\ln|\cos x| + C$ ✓

✓ 2. $\int \cot x \, dx = \ln|\sin x| + C$ ✓

3) $\int \sec x \, dx = \ln|\sec x + \tan x| + C$ ✓

✓ 4) $\int \csc x \, dx = -\ln|\csc x + \cot x| + C$ ✓

A: 1) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{-\sin x}{\cos x} \, dx$

✓ $= -\ln|\cos x| + C$

$= \ln|\cos x|^{-1} + C$

$= \ln \frac{1}{|\cos x|} + C$

✓ $= \ln|\sec x| + C$

3) $\int \sec x \, dx = \int \sec x \, \boxed{\frac{\sec x + \tan x}{\sec x + \tan x}} \, dx$ 1

$= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} \, dx$

$$= \ln|\tan x + \sec x| + C$$

Exp

$$\int_0^{\frac{\pi}{2}} \tan\frac{x}{2}\, dx$$

$u = \frac{x}{2}$

$du = \frac{1}{2}\, dx$

$2\, du = dx$

$x = 0 \Rightarrow u = \frac{0}{2} = 0$

$x = \frac{\pi}{2} \Rightarrow u = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}} \tan u\, (2)\, du$$

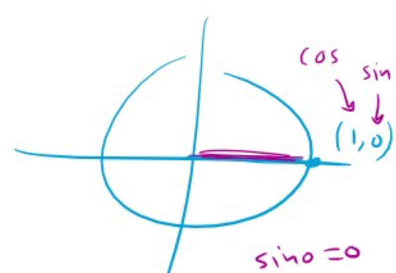
$$2\int_0^{\frac{\pi}{4}} \tan u\, du$$

$$(2)(-1)\ln|\cos u|\Big|_0^{\frac{\pi}{4}}$$

$$-2\left[\ln\left|\cos\frac{\pi}{4}\right| - \ln|\cos 0|\right]$$

$$-2\left[\ln\frac{1}{\sqrt{2}} - \ln 1\right]$$

$$-2\left[\ln 1 - \ln\sqrt{2} - 0\right]$$



$\sin 0 = 0$

$\cos 0 = 1$

$$-2\left[\underbrace{\ln 1}_{0} - \ln\sqrt{2} \quad - \quad 0\right]$$

$$-2\left[-\ln\sqrt{2}\right]$$

$$2\ln\sqrt{2} \quad = \quad \ln(\sqrt{2})^2 \quad = \quad \ln 2$$