Ch 5

Integration

1

Def: Antiderivative and integration

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$, $\forall x \in I$.

The set of all antiderivative of f is called the indefinite integral of f and is denoted by $\int f(x)\,dx$

Ex's

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$

$\int (3x^4 + 10x^{-3} + 10)dx = 3\frac{x^5}{5} + \frac{10x^{-2}}{-2} + 10x + C$

2) $\int \sin x\,dx = -\cos x + C$

3) $\int \cos x\,dx = \sin x + C$

4) $\int \sec^2 x\,dx = \tan x + C$

5) $\int \sec x \tan x\,dx = \sec x + C$

6) $\int \csc x \cot x\,dx = -\csc x + C$

7) $\int \csc^2 x\,dx = -\cot x + C$

Ex $\int \cos^2 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)dx = \int \frac{1}{2}dx + \frac{1}{2}\int \cos 2x\,dx$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

5.2 Definite integrals and areas: (2)

Integrals on a given interval is called definite integral and denoted by $\int_a^b f(x)dx$

Th Fundamental Theorem of Calculus:

① Suppose that f is continuous on $[a,b]$ and F is an antiderivative of f then

$$\int_a^b f(x)dx = F(b) - F(a)$$

② Suppose that f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, then F is continuous on $[a,b]$ and is differentiable on (a,b) and $\boxed{F'(x) = f(x)}$

Ex's

① $\int_2^3 (4x^3 - 1)dx = \frac{4x^4}{4} - x \Big|_2^3$

$= (3^4 - 3) - (2^4 - 2) = 78 - 14 = 64$

② $\left(\int_0^x \sin t\,dt\right)' = \sin x$

③ $\left(\int_a^{g(x)} f(x)dx\right)' = f(g(x)) \cdot g'(x)$

$\left(\int_1^{x^2} \frac{dt}{1+t^2}\,dt\right)' = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$

Area's

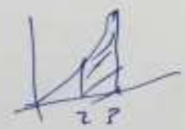
If $f(x) \geq 0$ is an integrable on $[a,b]$

then $\int_a^b f(x)dx$ is the area

enclosed between the curve $f(x)$ and the x-axis

Ex¹ Find the area enclosed between the following curves and the x-axis in the given intervals.

(1) $f(x) = x^2$, $x \in [2,3]$



$$A = \int_2^3 x^2 dx = \frac{x^3}{3}\Big|_2^3 = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

(2) $f(x) = 2x\sqrt{x^2+1}$, $x \in [0,1]$

$$\text{Area} = \int_0^1 2x\sqrt{x^2+1}\, dx \quad , \quad u = x^2+1$$

$$du = 2x\, dx$$

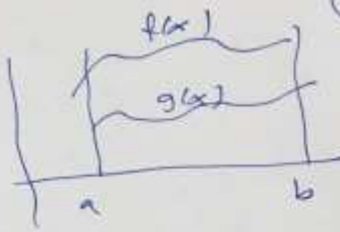
$$= \int_1^2 u^{\frac{1}{2}} du$$

$$= \frac{2}{3} u^{\frac{3}{2}} \Big|_1^2 = 2$$

$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 1\right) = \frac{2}{3}\left(2\sqrt{2} - 1\right)$$

If $f(x) \geq g(x)$ on $[a,b]$

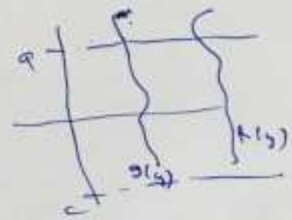
then the area enclosed between the two curves is



$$A = \int_a^b \left(f(x) - g(x)\right) dx$$

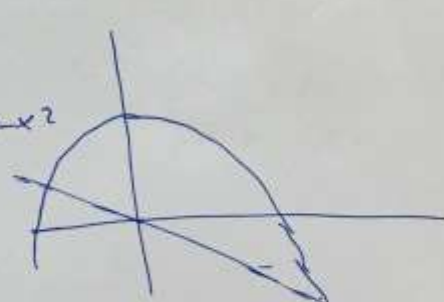
Similarly if $f(y) \geq g(y)$ on $[c,d]$

then the area between them is



$$A = \int_c^d \left(f(y) - g(y)\right) dy$$

Ex1 Find the area enclosed between the curves $f(x) = 2 - x^2$ and $g(x) = -x$



$\Rightarrow \quad 2 - x^2 = -x$

$0 = x^2 - x - 2$

$(x-2)(x+1) = 0$

$$A = \int_{-1}^{2} \left[(2-x^2) - (-x)\right] dx = \int_{-1}^{2} (2 - x^2 + x)\, dx = 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^{2} = \frac{9}{2}$$

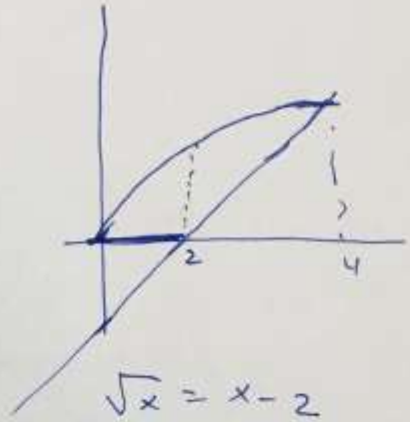
Ex2 Find the Area enclosed between $y=\sqrt{x}$, the x-axis and the line $y=x-2$

method 1

$$\int_0^2 \sqrt{x}\,dx + \int_2^4 [\sqrt{x} - (x-2)]\,dx$$

$$= \frac{x^{3/2}}{\frac{3}{2}}\Big|_0^2 + \left(\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x \Big|_2^4\right)$$

$$= \frac{10}{3}$$



$\sqrt{x} = x-2$

$x = (x-2)^2 = x^2-4x+4$

$\Rightarrow x^2 - 3x + 4 = 0$

$(x-4)(x+1) = 0$

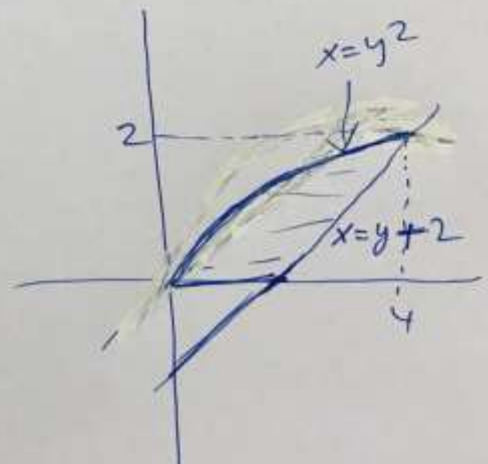
$x=4, \; x=-1$

method 2

$$A = \int_0^2 [(y+2) - y^2]\,dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_0^2$$

$$= \frac{10}{3}$$



$y = x-2$

$x = y+2$