Linear Systems and Equation

By: Rawan AlFares

lecture 1

1.1 systems of linear equations

Ex.1 $x_1+2x_2=5$
$2x_1+3x_2=8$

linear → Power of variables ≤ 1
\# of equations.
\# of unknowns

→ 2 equations.
→ 2 unknowns.

Sol:- 2 values for x_1 and x_2, that satisfies all equations.

* الحلول لازم تكون جواب صحيح لكل المعادلات.

is $x=\begin{bmatrix}1\\2\end{bmatrix}$ sol ?

$1+2(2)=5$ ✓
$2(1)+3(2)=8$ ✓. } substitue.

* أو ممكن نحلها بالحذف والتعويض

$$\begin{array}{l} [x_1+2x_2=5]\times -2 \\ +\ 2x_1+3x_2=8 \\ \hline \end{array}$$

$-x_2=-2 \longrightarrow \boxed{x_2=2}$

$\boxed{x_1=1}$

So $x=\begin{bmatrix}1\\2\end{bmatrix}$ → unique Solution.

* العمليات المسموحة على المعادلات ؛

1. أضرب بـ non-zero constant
2. أبدّل أماكن المعادلات
3. أضرب معادلة برقم وأجمعها للمعادلة الثانية

{ هون عادي لو كان الرقم صفر }

m x n linear system

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$
$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n$

$a_i \rightarrow$ coefficients

m :- # of equations → عدالمعادلات (rows)
n :- # of unkowns → عد المجاهيل (colomns)

* A solution to the system is a n-tuple (values $x_1, x_2 \dots x_n$) that satisfies all equations.

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $X = (x_1, x_2 \dots x_n)^T$.

EX. $x_1 - x_2 + x_3 = 2$
$2x_1 + x_2 - x_3 = 4$

2x3 system

unknowns x_1, x_2, x_3.

is $x = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ solution? (1) ✓ (2) ✓ so $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is a sol.

is $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ solution? (1) ✓ (2) so $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ is a sol

Sol: eq(1) + eq(2) = $3x_1 = 6$
$\boxed{x_1 = 2}$

$-x_2 + x_3 = 0$
$x_2 - x_3 = 0$

$\boxed{x_2 = x_3}$,

So any solution that has the form $x = \begin{bmatrix} 2 \\ x_2 \\ x_2 \end{bmatrix}$
, we have infinit number of solutions.

من القدس إلى أنت

Ex. $x_1+x_2=2$

$x_1-x_2=1$

$x_1 \quad =4$

3x2 system.

$x_1=4$, $\boxed{\begin{matrix} x_2=3 \\ x_2=-2 \end{matrix}}$ } no solution,

مافي قيمة لـ x_2 بتحقق
المعادلتين فلذلك بنحكي
no solutions.

} In Consistant.

m X n system

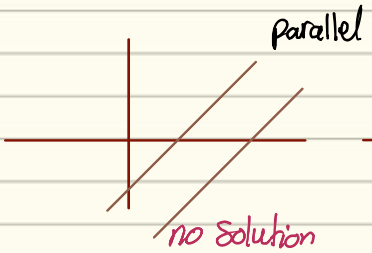
Consistant

- exactly one solution (unique Solution)
- infinite number of solution.

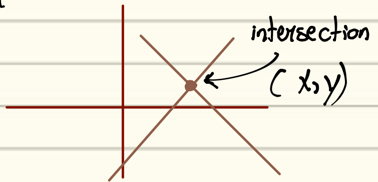
inConsistant ⇓ no solution.

هنسياً :-

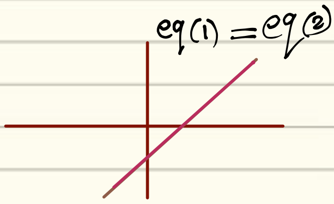
هنسياً :-
رسم أي خطين في
الهم 3 حالات :-



no solution
“inConsistant”



unique solution.
“Consistant”



infinite solution.
“Consistant”

Augemented matrix

Xlecture 2

Constants.

$$[\,A \mid b\,] = (A|b)$$

Coefficient

EX. $x_1 - x_2 + x_4 = 2$
$-x_1 + 3x_2 + x_3 + x_4 = 1$
$4x_1 - x_3 + 3x_4 = 5$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ -1 & 3 & 1 & 1 & 1 \\ 4 & 0 & -1 & 3 & 5 \end{array}\right] \longrightarrow -x_1 + 3x_2 + x_3 + x_4 = 1$$

elementary row operation

→ واذا ستخدمهم على ال system ما بأثر على الحل

1. Row operation 1 : Interchang two raws. $R_i \Longleftrightarrow R_j$
2. Row operation 2 : Multiply a row by non zero constant. $cR_i \;, c \neq 0$
3. Row operation 3 :- Replace a row by its sum with a multiple of another row. $(\alpha R_i + R_j \rightarrow R_j)$

EX. (a) $3x_1 + 2x_2 - x_3 = -2$
$x_3 = 3$
$2x_3 = 4$

Sol. $X = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$ back substitution

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{23} \end{bmatrix}$$ ← صيغتها العامة .

مثلث أصفار

* فاحنا لما بدنا نيجي نحل

بنقل ال Augemented matrix على نفس شكله

باستخدام ERO's .

Def:- two systems with the same unknowns, are called equivilant if they nave the same Solution sets.

equivilant System → Same variables
→ same Solution.

Remarks:-
gives an mxn system [A|b]
applying ERO's on [A|b] reduces an equivilant system [U|d], easy to Solve.

EX.
$3x_1+2x_2-x_3=-2$
$-3x_1-x_2+x_3=5$
$3x_1+2x_2+x_3=2$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ -3 & -1 & 1 & 5 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right] \xrightarrow[\frac{1}{3}R_1\to R_1]{\frac{1}{2}R_3\to R_3} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

مثلث من الأصفار
strict triangular form

$x_3=2$
$x_2=3$
$x_1+\frac{2}{3}x_2-\frac{1}{3}x_3=\frac{-2}{3}$
$x_1+2-\frac{2}{3}=\frac{-2}{3} \rightarrow \boxed{x_1=-2}$

$$X=\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$$

Remarks:-
An nxn system is said to be in Strict triangular form if in k^{th} equation, the Coeffiecint of $x_1 \dots x_{k-1}$ are all zero's and Coefficent of x_k is non zero.

هو يعني باختصار بكون مثلث من الاصفار تحت .

* إذا كان شكل النظام Strict triangular بكون الها فقط حل واحد

EX.

$$\begin{aligned}
x_1+x_2+x_3+x_4+x_5&=1\\
-x_1-x_2+x_5&=-1\\
-2x_1-2x_2+3x_5&=1\\
x_2+x_3+x_4+3x_5&=-1\\
x_1+x_2+2x_3+2x_4+4x_5&=1
\end{aligned}$$

Pivot

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1\\ -1 & -1 & 0 & 0 & 1 & -1\\ -2 & -2 & 0 & 0 & 3 & 1\\ 0 & 1 & 1 & 1 & 3 & -1\\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right] \xrightarrow[\substack{2R_1+R_3\to R_3\\ -R_1+R_5\to R_5}]{R_1+R_2\to R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 2 & 0\\ 0 & 0 & 2 & 2 & 5 & 3\\ 0 & 1 & 1 & 1 & 3 & -1\\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2\Longleftrightarrow R_4\\ R_3\Longleftrightarrow R_4 \end{array} \left[\begin{array}{ccccc|c} \textcircled{1} & 1 & 1 & 1 & 1 & 1\\ 0 & \textcircled{1} & 1 & 1 & 3 & -1\\ 0 & 0 & 1 & 1 & 2 & 3\\ 0 & 0 & 2 & 2 & 5 & 0\\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right] \xrightarrow[-R_3+R_5\to R_5]{-2R_3+R_4\to R_4} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1 & 3 & -1\\ 0 & 0 & 1 & 1 & 2 & 3\\ 0 & 0 & 0 & 0 & 1 & -6\\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{-R_4+R_5} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1 & 3 & -1\\ 0 & 0 & 1 & 1 & 2 & 3\\ 0 & 0 & 0 & 0 & 1 & -6\\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

the system is inconsistant.

Remarks-

A matrix M is said to be in Row echolen form if

1) the first non zero element in each non zero row is 1.

2) For each non zero row k, the number of leading zero's in row $k+1$ is greater than the number of leading zeros in row K.

3) if there are zero rows, they are below other rows.

EX.

$$\begin{bmatrix} \underline{1} & 2 & 0 & 1\\ 0 & 0 & \underline{\underline{1}} & 1\\ 0 & 0 & 0 & \underline{\underline{1}} \end{bmatrix}$$

(1) ✓
(2) ✓

REF

$$\begin{bmatrix} 0 & \underline{1} & 0 & 0 & 1\\ 0 & 0 & 0 & \underline{\underline{1}} & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(1) ✓
(2) ✓
(3) ✓

REF

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1\\ 0 & 0 & \underline{\underline{2}} & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

not REF

$$\begin{array}{c} 1\\ 2\\ 2\\ 5 \end{array} \begin{bmatrix} 0 & \underline{1} & 0 & 0 & 0 & 1\\ 0 & 0 & \underline{1} & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

not REF

* أقل رقم مش صفر بكل row لازم يكون 1.

* كل ما يزيد عدد ال rows لازم يكون عدد الأصفار أكثر من ال row اللي قبله

* إذا كان فيه صف كله أصفار لازم يكون آخر شي.

$$\begin{bmatrix} 1 & 0 & 2 & -3 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

not REF

Remark:-

Any matrix can be transformed to matrix in REF using ERO's.

EX. $A = \begin{bmatrix} \textcircled{1} & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$ pivot element

tranform A into REF.
(find the REF of A).

$$\xrightarrow[R_2 \Leftrightarrow R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 1 & 1 \end{bmatrix} \xrightarrow[-R_2 \to R_2]{2R_2+R_3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

↪ called REF of A

Remark:-

given $(A|b) \xrightarrow{ERO's} (U|b)$ in REF.
the system $(A|b)$ to $(U|b)$.
this method called Gauss Elemination method.

EX.
$$x_1 + 2x_2 + x_3 = 1$$
$$2x_1 - x_2 + x_3 = 2$$
$$4x_1 + 3x_2 + 3x_3 = 4$$
$$3x_1 + x_2 + 2x_3 = 3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right] \xrightarrow[\substack{-4R_1+R_3 \to R_3 \\ -3R_1+R_4 \to R_4}]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right] \xrightarrow[\substack{-R_2+R_3 \to R_3 \\ -\frac{1}{5}R_2}]{-R_3+R_4}$$

3x4 → under

leading ones

$$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 1 & 1 \\ 0 & \textcircled{1} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow x_3 = \alpha$$

REF

$$x_2 + \frac{1}{5}x_3 = 0$$
$$\boxed{x_2 = -\frac{\alpha}{5}}$$

$$x_1 + 2x_2 + x_3 = 1$$
$$x_1 - \frac{2\alpha}{5} + \alpha = 1$$
$$\boxed{x_1 = 1 - \frac{3\alpha}{5}}$$

$$X = \begin{pmatrix} 1 - \frac{3\alpha}{5} \\ -\frac{\alpha}{5} \\ \alpha \end{pmatrix}$$

بنطلع كل الأجوبة بدلالة x_3.

* the system is Consistant and has infinte number of sol.

* leading ones → الرقم (1) بعد الأصفار

* for each leading one their exist leading variables. → x_1, x_2

لكن بيضل x_3 ↓ فهو بعتبر free variable

* number of leading ones = number of leading variables.

* and rest of variables are called free variable.

* if the system has free variable, then it has infinite number of Sol.

* if their exist a row of the form $(0\ 0\ 0 \dots 0 \mid c \neq 0)$ then the system is inconsistent.

* if there are no free valiable, then the system has only one Solution.

GEM8- ملخص سريع :-

$$\begin{tikzcd}(A|b) \arrow[d, "\text{reduction to REF}"] \\ (U|b)\end{tikzcd}$$

أول اشي بتأكد من هاي

(1) if there is a row of the form $(0 \dots 0 \mid c \neq 0)$, then the system is inconsistent (No Solution).

(2) if not, then the system is consistent.

if there are free variables
⇓
infinite # of Solutions.
→ write the leading variables in terms of free variable.
⇓
then write the general form of the sol.

$X = (\quad)$

there are no free variable
⇓
only one solution (unique solution).
→ find it.
⇓
and then write the general form of the sol.

$X = (\quad)$

EX. $-x_2 - x_3 + x_4 = 0$

$x_1 + x_2 + x_3 + x_4 = 6$

$2x_1 + 4x_2 + x_3 - 2x_4 = -1$

$3x_1 + x_2 - 2x_3 + 2x_4 = 3$

$$\left[\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \xrightarrow[\substack{-3R_1 + R_4 \to R_4 \\ -R_2}]{-2R_1 + R_3 \to R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{array}\right]$$

$$\begin{array}{l} -2R_2 + R_3 \to R_3 \\ 2R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & -3 & -3 & -15 \end{array}\right] \xrightarrow[-\frac{1}{3}R_3 \to R_3]{-R_3 + R_4 \to R_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & +\frac{2}{3} & \frac{13}{3} \\ 0 & 0 & 0 & +1 & +2 \end{array}\right]$$

, no free variable

So, there is only one Sol.

$x_4 = 2$

$x_3 = \frac{9}{3} = 3$

$x_2 = -1$

$x_1 = 4$

$$x = \begin{bmatrix} 4 \\ -1 \\ 3 \\ 2 \end{bmatrix}$$

types of systems

1) overdeterminante system :- $m > n$
- unique Sol.
- infinite # of Sol.
- no Sol.

→ عادي بيقدر يطلع أي جواب

2) under determinant system :- $m < n$
- in Consistent
- infinite # of Sol.

→ مستحيل يطلع unique Sol.

3) Squar system :- $m = n$

Homogenuoes Systems:-

- over determinant
- under
- squar.

$a_{11}x_1 + a_{12}x_2 - - + a_{1n}x_n = 0$

Remarks:-

$X = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a Solution to Homogenuoes System.

→ Zero Solution. (trivial Sol.)

Any homogenuoes system is Consistant.

- only one Sol. (trivial Sol) Zero Sol.
- infinit number of Sol. (no Zero Sol.).

EX.

3×4

$$\left.\begin{aligned} -x_1 + x_2 - x_3 + 3x_4 &= 0 \\ 3x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 - x_2 - 2x_3 - x_4 &= 0 \end{aligned}\right\}$$

homogenuoes → trivial, infint

under detr → no Sol., infind.

infinits

$$\begin{bmatrix} +1 & -1 & +1 & -1 \\ 3 & 1 & -1 & -1 \\ 2 & -1 & -2 & -1 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 4 & -4 & 2 \\ 0 & 1 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow[-R_2+R_3]{\frac{1}{4}R_2} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} \underline{\underline{1}} & -1 & 1 & -1 \\ 0 & \underline{\underline{1}} & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

x_4 is free variable

$x_4 = \alpha$

$x_3 = \beta$

$x_4 = \alpha$

$x_2 - x_3 + \frac{x_4}{2} = 0 \longrightarrow X$

$x_2 = \beta - \frac{\alpha}{2}$

$x_1 - x_2 + x_3 - x_4 = 0$

$x_1 - \beta + \frac{\alpha}{2} + \beta - \alpha = 0$

$$\boxed{x_1 = \frac{-\alpha}{2}}$$

Def :- A matrix M is Called Reduced Row Echolen form (RREF) if

(1) M is in REF.

(2) each leading one is the only non zero element in it's Colomn.
this method Called Gauss Jordan elemination method.

EX. $\begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ RREF ✓ REF ✓

EX. $\begin{bmatrix} 0 & \overset{\checkmark}{1} & \overset{\checkmark}{0} & \overset{\times}{1} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

REF ✓
RREF ✗

* Any matrix can be transformed into a matrix in RREF

EX.
$$\left.\begin{aligned} -x_1 + x_2 - x_3 &= 1 \\ 3x_1 + x_2 - x_3 &= 0 \\ 2x_1 - x_2 - 2x_3 &= 2 \end{aligned}\right\}$$ use Gauss Jordan E.M.

$$\left[\begin{array}{ccc|c} +1 & -1 & +1 & -1 \\ 3 & 1 & -1 & 0 \\ 2 & -1 & -2 & 2 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 4 & -4 & 3 \\ 0 & 1 & -4 & 4 \end{array}\right] \xrightarrow{R_2 \Leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & -4 & 4 \\ 0 & 4 & -4 & 3 \end{array}\right]$$

$$\xrightarrow{-4R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 12 & -13 \end{array}\right] \xrightarrow{1/12\, R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & \frac{-13}{12} \end{array}\right] \xrightarrow{+R_2+R_1 \to R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & \frac{-13}{12} \end{array}\right] \xrightarrow[+3R_3+R_1]{+4R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & \frac{-13}{12} \end{array}\right]$$

$$X = \begin{bmatrix} -1/4 \\ -1/3 \\ \frac{-13}{12} \end{bmatrix}$$

EX.
$$\begin{aligned} -x_1 + x_2 - x_3 + 3x_4 &= 0 \\ 3x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 - x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

Ex. Consider the system

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 5 \\ x_1 - x_2 + a x_3 &= b. \end{aligned}$$

① for what values of a, b does the system have no Solution.

② 〃 〃 〃 〃 〃 〃 〃 〃 〃 〃 one Solution.

③ 〃 〃 〃 〃 〃 〃 〃 〃 〃 〃 inf # of Sol.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & a & b \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & a-1 & b-2 \end{array}\right]$$

① $a=1$ and $b \neq 2$

② $a \neq 1$

③ $a=1$ and $b=2$

1.3 Matrix Arathmatec

Matrix :- array of numbers or objects arranged in rows and Colomns denoted by A..

- A matrix A with m rows and n Colomns is called an m×n matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & \dots & \dots & \dots & a_{2n} \\ \vdots & & & & \\ a_{m1} & \dots & \dots & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

→ 1'st row

Colomn

m : # rows
n : # colomns

$\vec{a_i}$ = ith row = $(a_{i1} \; a_{i2} \; \dots \; a_{in})$

a_j = jth Colomn = $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

- a_{ij} is called entry in the i^{th} row, j^{th} Colomn
- m×n is called the Size, order, diminsion of A.
- for simplicity we write $A = (a_{ij})_{m\times n}$

Ex. $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & -1 & 1 & 3 \end{bmatrix}_{2\times 4}$

$a_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}_{2\times 1}$, $\vec{a_2} = [2 \; -1 \; 1 \; 3]_{1\times 4}$

$a_{23} = 1$

$A \rightarrow$ مصفوفة
$a_{ij} \rightarrow$ entry «مدخلة»

- **Colomn vector :** is an m×1 matrix

$A = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$, size = 3×1
Called :- Colomn vector
or colomn matrix.

- **row vector :-** is an 1×n matrix

$B = [1 \; 4 \; 3 \; 7]$, Size : 1×4
Called :- row matrix
row vector

Operations on matrices

1.Equality

$A = B$ iff ① they have the same size　　　　① نفس الحجم

② $a_{ij} = b_{ij}$　　　　② نفس المدخلات

ex. $\begin{bmatrix} 1 & 3 \\ 2x+1 & 3y^2 \end{bmatrix} = A$, $B = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

if $A = B$, find x , y

Ans :- $3 = 2x+1$ | $3y^2 = 9$

$x = 1$ | $y = \mp\sqrt{3}$

2.Addition and subtraction

* Same size.

$A_{m\times n}$, $B_{m\times n}$

$A \mp B = (a_{ij} \mp b_{ij})_{m\times n}$

ex. $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 5 \\ 6 & 7 \end{bmatrix}$

$\boxed{1}$ $A - C$ = undifined

$\boxed{2}$ $A+B = \begin{bmatrix} 5 & 2 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$\boxed{3}$ $2B - 3A = \begin{bmatrix} -5 & -6 & 5 \\ -12 & -15 & -18 \end{bmatrix}$

3.Scalar multiplication

↳ Scalor ≡ Real number

$\alpha A = \alpha\, a_{ij}$

ex. $A = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix}$, find $2A = \begin{bmatrix} -4 & 2 \\ 0 & 10 \end{bmatrix}$

4.Zero matrix

is a matrix whose enteries all zero.

(O)

ex. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times1}$, $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{1\times3}$

من للقدس
إلا أنت

Some properties of addition and scalar multiplication

1. $\alpha(A+B) = \alpha A + \alpha B$
2. $(\alpha B)A = \alpha(BA) = B(\alpha A)$
3. $A+B = B+A$ Commutative law $AB \neq BA$
4. $(A+B)+C = A+(B+C)$ Assotiative law $\underset{m\times n}{A}(\underset{n\times r}{B}\underset{r\times s}{C}) = (\underset{m\times n}{A}\underset{n\times r}{B})\underset{r\times s}{C}$ n×s m×r
5. $A+0 = 0+A = A$
   ↳ zero matrix
6. $A-A = A+(-A) = 0$
   ↳ additive inverse of A.
7. $A(B+C) = AB+AC$, $(A+C)B = AB+CB$ distributive law
8. $(\alpha+\beta)A = \alpha A + \beta A$

properties of the transpose

* let A be m×n, matrix, we define the transpose of A as the matix C, where $A^T = C = c_{ij}$, where $c_{ij} = a_{ji}$.

ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$, find $A^T = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$

* باختصار الكولوم بصير row
row بيصبح Colomn.

$A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & -1 & 6 \\ 5 & 6 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} 3 & 4 & 5 \\ 4 & -1 & 6 \\ 5 & 6 & 2 \end{bmatrix}$

they are equal → symmetric Matrix.

* An n×n matrix is Called Symmetric if $A^T = A$

$A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & -1 & 6 \\ 5 & 6 & 2 \end{bmatrix}$

main diagonal.

symmetric

properties for the transpose

1) $(A^T)^T = A$, double transpose to the matrix, giv'n you the original matrix.

2) $(A \mp B)^T = A^T \mp B^T$

3) $(\alpha A)^T = \alpha (A^T)$

4) $(AB)^T = B^T A^T$

5) if $A_{n \times n}$ is symmetric, $B_{n \times n}$ is symmetric then $A+B$ is Symmetri

6) if $A_{n \times n}$ is symmetric, then αA is symmetric.

* For any matrix A, it can be multiplied with A^T, and the produce matrix is square.

* An nxn matrix A, is said to be skew-symmetric if $A^T = -A$

EX. let A be mxn matrix, Let $C = A.A^T$, Is C symmetric?

$C = AA^T$

$C^T = (AA^T)^T$

$C^T = (A^T)^T A^T$

$C^T = AA^T$

$C^T = C$

so, C is symmetric , Also A^TA is symmetric

EX. let A and B be symmetric, then $H = AB - BA$ is symmetric

A is symmetric ⇒ $A = A^T$

B is symmetric ⇒ $B = B^T$

we want to check if H is symmetric or not.

$$(AB-BA)^T = (AB)^T - (BA)^T$$

$$= B^T A^T - A^T B^T$$

$$= BA - AB \neq H$$

So its not symmetric.

Skew Symmetric

A matrix is skew symmetric , $A^T = -A$

ex. $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$

$-A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$, $-A = A^T \Rightarrow$ skew symmetric

$\Rightarrow$ on diagonals equals zero.

$\Rightarrow$ outside diagonal each number have the opposite sign to the same magnitude number.

Q.16 | 1.3

Prove that for any skew Symmetric the diagonal equals zero.

if A is skew symmetric , then $A^T = -A$

for $A^T = a_{ji} = -(a_{ij}) = -A$

for $(i=j) \Rightarrow a_{ii} = -a_{ii} \Rightarrow \frac{2a_{ii}}{2} = \frac{0}{2} \Rightarrow \boxed{a_{ii} = 0}$

$+a_{ii}$ $+a_{ii}$

Q. let A be nxn matrix, let $C = A - A^T$, $B = A + A^T$

Show what type of symmetric are C and B.

$C = A - A^T$

$C^T = (A - A^T)^T$

$C^T = A^T - (A^T)^T$

$C^T = A^T - A = -(A - A^T)$

$= -C$

so, C is skew symmetric.

$B^T = (A + A^T)^T$

$= A^T + (A^T)^T = \underline{A^T + A}$

$B^T = B$

so B is symmetric.

ex. if A is an $m \times n$ Matrix, the A^TA and AA^T both symmetric

※ prove :-

we want to show that $A^TA = A^TA$

$$(A^TA)^T = A^T.(A^T)^T$$
$$= A^T . A$$

$$(A^TA)^T = A^T.(A^T)^T$$
$$= A^T . A$$

Matrix multiplication and linear system

should be the same

$A_{m\times n}$, $B_{n\times K}$ then $AB = C_{m\times K}$

size of the new matrix.

*

* $C_{ij} = \sum_{n=1}^{n} a_{ik} b_{kj}$

new size

Ex. $A = \begin{bmatrix} 1 & 3 \\ 6 & -1 \end{bmatrix}_{2\times 2}$, $B = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -3 \end{bmatrix}_{2\times 3}$

1. $AB = \begin{bmatrix} 1 & 8 & -7 \\ 6 & 29 & 15 \end{bmatrix}_{2\times 3}$

2. $BA =$ $2\times 3 \quad 2\times 2$ undifined

$BA \neq AB$

matrix multiplication is not commutative.

linear system

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \vdots & & \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & & \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Cooficient matrix — unknowns — Constants

$$A . X = b$$

another way :-

$$Ax = \begin{bmatrix} a_{11} x_1 \\ a_{21} x_1 \\ \vdots \\ a_{m1} x_1 \end{bmatrix} + \begin{bmatrix} a_{12} x_2 \\ a_{22} x_2 \\ \vdots \\ a_{m2} x_2 \end{bmatrix} + \dots\dots + \begin{bmatrix} a_{1n} x_n \\ a_{2n} x_n \\ \vdots \\ a_{mn} x_n \end{bmatrix} = b$$

$$Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + X_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = b$$

$$AX = X_1 a_1 + X_2 a_2 + \dots + X_n a_n = b .$$

where a_i are the coloms of A

ex. write a matrix form:-

$4x_1 + 2x_2 + x_3 = 1$
$5x_1 + 3x_2 + 7x_3 = 2$

Ans :-

$$\begin{bmatrix} 4 & 2 & 1 \\ 5 & 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$x_1 a_1 + x_2 a_2 + x_3 a_3 = b$ → linear compination

Remarks:-

An mxn System Can be written in the following forms :-

1. $[A|b]$ Augemented matrix
2. $Ax = b$ matrix form
3. $x_1a_1 + x_2a_2 + \dots + x_na_n = b$. Colomns of A form.

Ex. $b = \underset{x_1}{2}a_1 + \underset{x_2}{3}a_2 + \underset{x_3}{4}a_3$, find the Solution :-

Ans : $(2, 3, 4)$.

ex. give that $A_{3\times3}$, $Ax = b$, $b = 4a_1 - 6a_2 + 3a_3$
Then $\begin{bmatrix} 4 \\ -6 \\ 3 \end{bmatrix}$ is the solution.

ex. given that $A_{3\times3}$, $a_3 = a_1 - a_2$, then $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is a Sol of $Ax = ??$.

$Ax = b$ $\qquad\qquad$ $Ax = b$

$a_1 - a_2 - a_3 = 0$ ↗

$\underset{1}{x_1}a_1 + \underset{-1}{x_2}a_2 + \underset{-1}{x_3}a_3 = 0$ → $x = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ yes it's a Sol.

Remark :- 1. x_0 is a Sol. of $Ax = b$, iff $Ax_0 = b$

2. if x_1 and x_2 are Sol. of $Ax = b$, then $\alpha x_1 + \beta x_2$ is a Sol. iff $\alpha + \beta = 1$.

Prove :- x_1 , x_2 are sol. of $Ax=b$

Ans : $\boxed{Ax_1=b}$, $\boxed{Ax_2=b}$

$\alpha x_1+\beta x_2$ should be also , A Sol.

$A(\alpha x_1+\beta x_2)=b$

$\alpha\boxed{Ax_1}+\beta\boxed{A\,x_2}=b$

$\alpha b+\beta b=b$

$(\alpha+\beta)b=b$

iff $\boxed{\alpha+\beta=1}$ #

Prove : if x_1 and x_2 are sol. for $Ax=0$, then $\alpha x_1+\beta x_2$ $=0$ is a Sol. of $Ax=0$, $\forall\alpha,\beta\in\mathbb{R}$.

$Ax_1=0$, $Ax_2=0$

$A(\alpha x_1+\beta x_2)=0$

$\alpha Ax_1+\beta Ax_2\overset{??}{=}0$

$\alpha(0)+\beta(0)\overset{??}{=}0$

$=0$

#

ex. $\begin{bmatrix}2\\5\end{bmatrix}=2\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$

↳ linear Compination from two other vectors.

consistentcy theorem

A linear System $Ax=b$ is Consistant, iff b can be written as a linear Combination of the columns of A.

ex. $x_1+2x_2=1$
$2x_1+4x_2=1$

$c_1\begin{bmatrix}1\\2\end{bmatrix}+c_2\begin{bmatrix}2\\4\end{bmatrix}\overset{??}{=}\begin{bmatrix}1\\1\end{bmatrix}$

→ No. Sol. inconsistent.

Euclidean n-space

$\equiv \mathbb{R}^n : \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} :- x_i \in \mathbb{R} \right\}$ All $n \times 1$ matricies with Real enteries.

$\mathbb{R}^{n \times 1}$

ex. $X \in \mathbb{R}^3 \longrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ x_1, x_2, x_3 \in \mathbb{R}$

$\mathbb{R}^{1 \times n}$ = All $1 \times n$ entries with real entries

Colomns

ex:- $X \in \mathbb{R}^{1 \times 3}$, $X = [\, x_1 \ x_2 \ x_3 \,] \in \mathbb{R}$

Remark:- $\mathbb{R}^{m \times n}$:- All matricies are Real number entries. $m \times n$

ex. $a_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \quad a_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$

① find a linear Combination of a_1, a_2, a_3.

there are infinite number of combination,

$$2a_1 + 3a_2 - 4a_3 = 2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - 4\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 18 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 15 \\ 18 \\ 0 \end{bmatrix}$ is a linear Combination.

② is $\begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix}$ a linear compination :- → means Solve the System

Consistant → yes linear combination

inconsistant → No, Not linear compination.

$$\begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$$

$$\begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \longrightarrow \circledast$$

Solve the system !!

$\begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix}$ is linear compination iff $\circledast$ is Consistant.

Remarks:-

① the system is Consistant iff b can be written as a linear Combination of colomns of A.

② if $x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$, is a Sol to $Ax=b \iff b = x_1a_1+x_2a_2+\dots x_na_n$

Ex. let A be 5x3 matrix and $b=a_1+a_2=a_2+a_3$
what can we conclude about the number of sols of $Ax=b$

Ans:- $b = 1.a_1+1.a_2+0.a_3 \longrightarrow X=\begin{bmatrix}1\\1\\0\end{bmatrix}$, then Consistant

$b=0.a_1+1.a_2+1.a_3 \longrightarrow X_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$ another sol.

So, it has infinite number of Sol.

ex. if y and z are Sol. to $Ax=b$

is $(y+z)$ a Sol to A.

$Ay=b$, $Az=b$

$A(y+z)=b$

$Ay+Az=b$

$b+b\neq b$

$(y+z)$ not Sol.

is $(\frac{1}{4}y+\frac{3}{4}z)$ a Sol. ?

$A(\frac{1}{4}y+\frac{3}{4}z)\overset{??}{=}b$

$=\frac{1}{4}b+\frac{3}{4}$

$=b$ # yes Sol.

$\boxed{\alpha+\beta=1}$ → for non homogenuoes

α,β any number for homogenuoes numb

ex. if y is a Solution $Ax=b$, z is a Solution for $AX=0$, is $y+z$ Sol for $Ax=b$.

Ans:- $A(y+z)\overset{??}{=}b$

$0+b=b$ # is a sol. for the non homogenuoes system.

Q.12 → 1.3

1.4 Speacial matrices

* Zero matrix $O = O_{m\times n}$

$$O = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & & \end{bmatrix}$$

- $A + O = A$
- $\underset{m\times n}{A} \cdot \underset{n\times r}{O} = \underset{m\times r}{O}$, also $O \cdot A = O$
- let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \neq 0$

but $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $AB = O$

$A \neq 0$, $B \neq 0$ but $AB = 0$.

real numbers

$x, +, -$

(zero) (Identity)

$0 + x = x$

$0 \cdot x = 0$

real numbers

$xy = 0$

$\Downarrow$

$x = 0$ or $y = 0$

impossible

$x \neq 0, y \neq 0$

$xy = 0$

not possible

* if A, B matricies and $AB = 0$,

$\cancel{\Rightarrow}$ $A = 0$ or $B = 0$.

* يعني اذا ضربنا مصفوفة A مع B ونتج عنّا مصفوفة صفرية ، ما بنقدر نستنتج أي اشي بخصوص A أو B.

* Identity matrix :- $I_n = (\alpha_{ij})_{n\times n}$

- squar matrix.

when $\alpha_{ij} = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases}$

$$I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n\times n} , \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

always (1)

always zero

real number

$1 \in \mathbb{R}$

* $1 \cdot x = x \cdot 1 = x$

* for any $x \neq 0$,

$x \left[\frac{1}{x}\right] = 1$, $\frac{1}{x} \in \mathbb{R}$

↳ multiplicative inverse of x.

Properties for the Identity matrix :-

1. $A_{m\times n} I_n = A_{m\times n}$ and $I_m A_{m\times n} = A_{m\times n}$

* Def: let $A = (a_{ij})_{n\times n}$, we say A is non singular (invertible) if there exists a matrix, $B = (b_{ij})_{n\times n}$, such that $AB = I = BA$

* إذا لاقيت مصفوفة (A) بضربها من اليمين واليسار بتعطيني I , فهون A بتكون non singular

So, if B exist is called an iverse of A.

$A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$ is B inverse of A.?

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

so B is inverse of A, and so A is non singular.

* In general : $AB \neq BA$, only in inverses, must $AB = BA = I$

* if C, D are $n\times n$-matricies and $CD = I = DC$:

① C and D are non singular.

② inverse of C is D and vice versa.

Ex. $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$. Is A non singular?
try to find $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ s.t $AB = I = BA$?

$A.B = I$

$$\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2b_1 + b_3 & 2b_2 + b_4 \\ 6b_1 + 3b_3 & 6b_3 + 3b_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* if the system is inconsistent, then it's singular,
if not, it's nonsingular

$$\left.\begin{matrix} 2b_1 + b_3 = 1 \\ 6b_1 + 3b_3 = 0 \end{matrix}\quad , \quad \begin{matrix} 2b_2 + b_4 = 0 \\ 6b_2 + 3b_4 = 1 \end{matrix}\right\}$$

non singular → has inverse
singular → has no inverse

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 6 & 0 & 3 & 0 & 0 \\ 0 & 6 & 0 & 3 & 1 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 6 & 0 & 3 & 1 \end{array}\right] \rightarrow \text{inconsistant}$$

so it's singular.

so $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ is singular, has no inverse.

Remarks:-

* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $ad - bc \neq 0$

then A is nonsingular and inverse of A is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* if $ad - bc = 0$, then A is singular.

Ex. $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ nonsingular or no?

$2 \times 5 - 3 \times 0 = 10 \neq 0$ so yes, its nonsing

and the inverse is

$$\frac{1}{10}\begin{bmatrix} 5 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-3}{10} \\ 0 & \frac{1}{5} \end{bmatrix}$$

Remarks:-

1. if $A_{n\times n}$ is nonsingular, then inverse of A is unique.

Proof:- Assume A is nonsingular (has inverse), to show inverse of A is unique, Assume B and C are both inverses of A

* $B = C$.

B is inverse of $A \rightarrow AB = I = BA \rightarrow$ ①

C is inverse of $A \rightarrow AC = I = CA$ ②

$$\boxed{B} = BI \overset{②}{=} B(AC) = (BA)C \overset{①}{=} IC = \boxed{C}$$

$B = C$ \# unique inverse

Remarks: 1 if A is non singular we denote it by A^{-1}

* $A.A^{-1} = I = A^{-1}.A$

2 if A, B are non singular nxn matracies, then

* AB = is also non singular
* $(AB)^{-1} = B^{-1}.A^{-1}$

Proof.

Assume A, B are non singular [A^{-1} exists, B^{-1} exists]

Ans: ① $(AB).(B^{-1}A^{-1}) = A.(B.B^{-1}).A^{-1} = A.I.A^{-1} = A.A^{-1}$
$= I$

② $(B^{-1}A^{-1}).(AB) = B^{-1}.(A.A^{-1}).B = B^{-1}.I.B = B^{-1}.B = I$

so AB is non singular and $(AB)^{-1} = B^{-1}A^{-1}$.

3 if $A_1, A_2, A_3 ....., A_K$ are non singular, nxn matricies, then $A_1.A_2.A_3.......A_K$ is also non singular and
$(A_1 A_2 .... A_K)^{-1} = A_K^{-1} A_{K-1}^{-1} ... A_2^{-1}.A_1^{-1}$

4 if A is nxn-matrix, then $\underbrace{A.A....A}_{K\text{ times}} = A^K$, $K \in Z^+$

TRUE OR FALS QUESTIONS

1] if A, B are non singular, $n \times n$-matricies, then $A+B$ is non singular. F

Counter example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A+B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 1 - 1 = 0$ so it singular

2] the sum of two singular matricies is singular. F

Counter example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = 2 - 0 = 2 \neq 0$,

3] if A is singular, B is non singular, then $A+B$ is singular False

4] if A is singular, B is non singular, then $A+B$ is non singular F

5] if A, B are $n \times n$ matricies, then $A^2 - B^2 = (A-B)(A+B)$ F

6] if A, B are $n \times n$ matricies, $AB = 0$, then $A = 0$, or $B = 0$ F

7] if $A^2 = 0$, then $A = 0$ F. Counter example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = 0$

8] $AB = AC$, then $B = C$ F

multiply by A^{-1} from left $\rightarrow$ $A^{-1}AB = A^{-1}AC$

$B = C$

لكن هون كيف نضرب بـ A^{-1}

فاحنا مش عارفين عنها

اشي .

Need not exist.

9] if A is nonsingular and $AB = AC$, then $B = C$ T

10] if A is $n \times n$ matrix and $A^2 = A$, then $A = I$ F

11] if A is $n \times n$ matrix and $A^2 = A$, then $A = 0$ or $A = I$. F counter example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

12] if $A_{n \times n}$ Such that $A^2 = A$, then $(A+I)$ is nonsingular and $(A+I)^{-1} = I - \frac{1}{2}A$. T

Proof #.

$(A+I).(A+I)^{-1} = I$ $\Leftarrow$

$=(A+I).(I-\frac{1}{2}A)$

$=(A.I)+(A.(-\frac{1}{2}A))+(I.I)+(I.(-\frac{1}{2}A))$.

$= A + -\frac{1}{2}A^2 + I + -\frac{1}{2}A$

$= A - \frac{1}{2}A + I - \frac{1}{2}A$

$= \cancel{A - A} + I$

$= I$ #

so, $(A+I)$ is nonsingular
and $(I-\frac{1}{2}A) = (A+I)^{-1}$

Also, $(I-\frac{1}{2}A).(A+I)$
$= I$ #

13] if $A_{n\times n}$, $A^2=0$, then $I-A$ is nonsingular and $(I-A)^{-1}$
$= I+A$ True

$(I-A).(I-A)^{-1} = I$

$\Rightarrow (I-A).(I+A)$

$= I + \underbrace{\cancel{A - A}} - \cancel{A^2}$

$= I$ # so, $A+I$ is nonsingular
and $(I-A)^{-1} = I+A$.

$I+A$ is nonsingular and $(I+A)^{-1} = I-A$

Also, $(I-A)^{-1}.(I-A)$
$= I$
check !! #

an example of $A^2=0$, and $A\neq 0$

$A = \begin{bmatrix}0&1\\0&0\end{bmatrix}$, $A.A = \begin{bmatrix}0&1\\0&0\end{bmatrix}.\begin{bmatrix}0&1\\0&0\end{bmatrix}$

$= \begin{bmatrix}0&0\\0&0\end{bmatrix}$ #

Remark :-

if A is nonsingular, then A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

Proof :- Assume A is nonsingular

Consider, $A^{-1} \cdot \underset{A}{\square} = I$

$A^{-1} \cdot \underset{(A^{-1})^{-1}}{\square} = I$

then $A = (A^{-1})^{-1}$

#

Elementary Matricies

A matrix E is called an elemantary matrit, if it's obtained from the identity (I_n), by performing exactly one row operation.

There are 3 types :- **If multiplying from (left row operations)**

- interchanging any two rows.

  $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$

  elemantry matrix.

  interchange row 1 with row 2.

- mulliplying any row by non zero constant.

  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

  elemantary matrix

  $2R_3$

- adding a multiple of one row to another row.

  $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

  elemantry matrix

  $(3R_3 + R_1)$

Remark :-

1. multiplying a matrix A from left by an elemantary matrix is the Same as performing a row operation on A of the same type.

ex. $E = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, $EA = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 & -2 \\ 4 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -2 \\ 16 & 0 & 28 \end{bmatrix}$

type II $[4R_2]$ → فقط نضرب 2^{nd} Row by (4).

فالعملية الي عملناها ←
على E رح تتكون نفسها الي
نعملها على A وهي نفسها
الجواب .

2. multiplying a matrix A from Right by an elemantary matrix is the Same as performing Column operation on A of the same type.

AE

ex. $\begin{bmatrix} 3 & -2 & 0 \\ 1 & 5 & 4 \\ 2 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -14 & 0 \\ 1 & 35 & 4 \\ 2 & 0 & 1 \end{bmatrix}$

elemantary matrix $[7R_2]$

NOTE :- I is nonsingular.

if E is elemantary Matrix then E is nonsingular, and E^{-1} is elemantary of the same type.

※ Proof :- let E be elemantary matrix :-

$\boxed{1}$ if E is of type 1. $(R_i \rightarrow R_j)$.

$\underbrace{E}\, . \, E =$

العملية التي قمنا بها على E, ستطبق على E.

$I \xrightarrow{Ro's} E$

$E \xrightarrow{Ro's} I$

$E = E^{-1}$

* only in type 1, the inverse of E equals itself.

ex. $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ interchange R_1 with R_2

$$E.E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

R_1, R_2

$\boxed{2}$ if E is of type 2. $(\alpha R_i \, , \, \alpha \neq 0)$

$I \xrightarrow{\alpha R_i} E$

let F be the matrix obtained from I by the Row operation $\frac{1}{\alpha} R_i$.

$\underset{\alpha}{E} . \underset{\frac{1}{\alpha}}{F} = I = F.E$

E is nonsingular and it's inverse equals F, elemantary with the same type.

ex. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\boxed{3}$ if E is of type 3. $(\alpha R_i + R_j)$

$$I \xrightarrow{cR_i+R_j} E$$

let F be the matrix obtain from I by $-cR_i+R_j$

$$E.F = I = F.E$$

+c -c

ex. $E = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ex. $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Row Equivalent

B is row equivalent to A if there exists finite sequence of elementary matricies, $E_1 E_2 \dots E_k$, Such that $B = E_k E_{k-1} \dots E_1 A$

In another words :-

B is row equivilant to A, if B can be obtained from A by finite of row oper.

ex. $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{bmatrix}$

A) is B row equivalent to A ?

B) is C row equivilant to A ?

c) find elemantary Matricies such that EA = B.

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Ans :- A). $B = EA$

$= A \xrightarrow{1R_1+R_3} B$

So, yes, $B \cong A$

d) find elemantary matricies such that FB = C ?

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

B) $C = E_2 E_1 A$

$$A \xrightarrow[E_1]{1R_1+R_2} B \xrightarrow[E_2]{R_2-R_3} C$$

E) is C row equivilant to A ?

$C = E_1 E_2 A$

elementary matricies.

Remark :-

1. if A is row equ. to B then B is row eq. to A.

2. if $A \hat{=} B$
   $B \hat{=} C$
   then $A \hat{=} C$
   ↳ Row equivilant.

※ Proof :

$A \hat{=} B$

↳ $B = E_k E_{k-1} \dots E_2 E_1 A$

E_k^{-1} E_k^{-1}

$$E_1^{-1} E_2^{-1} \dots E_{k-1}^{-1} E_k^{-1} B = A$$

$$A = \underbrace{E_1^{-1} E_2^{-1} \dots E_k^{-1} E_1^{-1}}_{\text{elemantry Matrix}} B$$

$$A \hat{=} B, \quad B \hat{=} C \Longrightarrow \begin{array}{ll} B = E_k E_{k-1} \dots E_2 E_1 A & \longrightarrow (1) \\ C = F_k F_{k-1} \dots F_2 F_1 B & \longrightarrow (2) \end{array}$$

Substitute (1) in (2)

$$C = \underbrace{(F_k F_{k-1} \dots F_2 F_1)(E_k E_{k-1} \dots E_2 E_1)}_{\text{elemantry matricies}} A$$

so $C \hat{=} A$
since C can be obtained from A. so C is row equivilant to A.

* Two Augemented Matricies $[\vec{A}|\vec{b}]$, $[\vec{B}|\vec{c}]$ are row equivilant $\Longleftrightarrow$ $A\vec{x} = \vec{b}$ & $B\vec{x} = \vec{c}$

Theorem

let A be an $n \times n$ matrix. then the following statment are equivilant.

(a) A is nonsingular.

(B) $Ax = 0$, has only the trivial Sol. ($x=0$ is the Sol)

(c) A is row equivilant to I_n.

(a) ⇔ (b)
(b) ⇔ (c)
(c) ⇔ (a)

proof: (a) ⇔ (b)

let y be asolution for $Ax=0$,

then $Ay=0$

$$A^{-1}Ay = A^{-1}.0$$

$Iy=0 \rightarrow y=0$, So, it has only the Zero Sol.

Proof (b) ⇔ (c)

- suppose $Ax=0$, has only one sol.
- suppose that A is not row equivilant to I.

So the RREF to A has free Variables, and $Ax=0$, has infinity many sol.

which is Contradiction.

proof (c) to (a).

suppose that A is row equivilant to I_n, so there exist finite sequence of elementary matricies, $E_1 E_2 \dots E_k$ such that

$$E_k E_{k-1} \dots E_2 E_1 I = A$$

$$A = E_k E_{k-1} \dots E_2 E_1$$

$$A^{-1} = E_1^{-1} E_2^{-1} \dots E_{k-1}^{-1} E_k^{-1}$$

So A is nonSingalar.

Method to find A invers

if $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$. find A^{-1} (if exist).

$$[A \mid I] \xrightarrow{ERO's} [I \mid A^{-1}].$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} -R_2+R_1 \to R_1 \\ 2R_2+R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right] \xrightarrow{-2R_3+R_2} \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \;\middle|\; \underbrace{\begin{matrix} 1 & -1 & 0 \\ 4 & -3 & -2 \\ -2 & 2 & 1 \end{matrix}}_{A^{-1}} \right]$$

Remark :-

if In the process of performing row operations on $[A \mid I_n]$ one row of A reduced the A reduced to a zero row. then A^{-1} doesn't exist

ex. $A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & -1 \\ -2 & 2 & -6 \end{bmatrix}$, find A^{-1} (if any)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ -2 & 2 & -6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[2R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \end{array}\right]$$

A has no inverse (singular).

Remark :- $Ax = b$

- infinite # of sol.
- unique sol (zero sol.)
- no sol.

$A X = 0$

- infinite # of sol.
- only zero sol.

Solve x by using $X = A^{-1}.b$

let A be a nonsingular matrix and it has a unique sol.

$$Ax = b$$

$$A^{-1}.Ax = A^{-1}.b$$

$$x = A^{-1}.b \longrightarrow$$ So $Ax=b$. has aunique Sol.

✗ to show that A is nonsingular.

assume that $Ax=0$ has infinite number of Solution and

A is nonsingular.

$$A^{-1}AX = A^{-1}.0$$

$$IX = 0$$

$$\boxed{X=0} \longrightarrow$$ one sol. (contradiction).

So A is nonsingular matrix and has only the zero sol.

ex. Solve the system $\left\{\begin{array}{l} x_1 + x_2 + 2x_3 = -2 \\ \quad\;\; x_2 + 2x_3 = 3 \\ 2x_1 + \quad\; + x_3 = 0 \end{array}\right\}$, $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

$\hookrightarrow A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -2 \\ -2 & 2 & 1 \end{bmatrix}$ من المثال السابق

$$X = A^{-1}.b$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -2 \\ -2 & 2 & 1 \end{bmatrix} . \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -5 \\ -7 \\ 10 \end{bmatrix}$$, $x_1 = -5$, $x_2 = -7$, $x_3 = 10$ ⟶ unique Sol. ⇓ + square لأنه nonsingular.

Remarks:- 1. $A_{n\times n}$ is non singular $\Leftrightarrow$ $\boxed{Ax=0}$, has only one sol.

2. $A_{n\times n}$ is singular $\Leftrightarrow$ $Ax=0$ has infinite ✗ of sol.

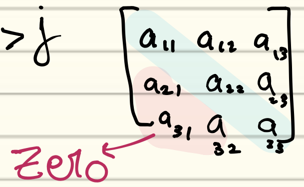
3. $A_{n\times n}$ is non singular $\Leftrightarrow$ $Ax=b$ has aunique Sol.

4. $A_{n\times n}$ is Singular $\Leftrightarrow$ $Ax=b$ either no Sol. or inf. ✗ of sol.

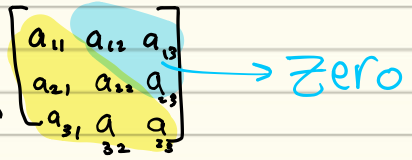
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Diagonal And Triangular Matrices

① upper triangular :- $a_{ij} = 0$, for $i > j$



② lower triangular : $a_{ij} = 0$, $i < j$



③ Triangular, if it's either upper or lower triangular.

④ diagonal : upper and lower. , $a_{ij} = 0$, for $i \neq j$

ex. $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 4 & 3 \end{bmatrix}$ lower triangular.

upper triangular.

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

lower & upper
& diagonal.

& triangular

$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

lower
upper
traingular
diagonal.

LU Factorization

1. given a system $Ux=b$, U is upper triangular.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Ans:- $x_3 = 3$
$x_2 = -10$
$x_1 = 25$

2. given a system $Ux=b$, U is lower triangular.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

Ans: $x_1 = 3$
$x_2 = -1$
$x_3 = 0$

* given $Ax=b$ and $A=LU$

$L(UX) = b$

L , u > معطى في السؤال

$\overset{\checkmark}{L} y = \overset{\checkmark}{b} \leftarrow$ معطى

نجيب y

$y = UX$

$\overset{\checkmark}{y} = U \boxed{X} \longrightarrow$ نجيب X

أوجدناه ← ↳ معطى

وهكذا نكون أوجدنا حل السؤال.

ex.

$A = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$, $A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$.

$Ax = b$

$LUx = b$

$Ly = b \rightarrow ①$

$Ux = y$

Solve $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$y_1 = 1$

$-2y_1 + y_2 = 2 \rightarrow y_2 = 4$

$3y_1 - 2y_2 + y_3 = 3 \rightarrow y_3 = 8$

$y = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$

Solve $Ux = y$

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} ,$$

$x_3 = 4$

$x_2 = -\frac{4}{3}$

$x_1 = +\frac{17}{6}$

$\Bigg\} \quad x = \begin{bmatrix} 17/6 \\ -4/3 \\ 4 \end{bmatrix}$

من القدس إلى أنت

How to find L ,U $A = LU$.

$A = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$, find L, U, s.t $A = LU$

(not always possible).

when it's possible ?

↳ if A can be reduced to an upper triangular matrix U using only row operation III.

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix} \xrightarrow[-3R_1+R_3]{2R_1+R_2} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & -6 & 12 \end{bmatrix} \xrightarrow{2R_2+R_3} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} = U$$

⇓

بدي أحولها ل

(U)

من خلال استخدام

Row operation 3

only.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

* Remainder :
the inverse of Row op. III

$(cR_i + R_j) \longrightarrow -cR_i + R_j$

	Invers	
$2R_1+R_2$	$-2R_1+R_2$	→ L_{21}
$-3R_1+R_3$	$+3R_1+R_3$	L_{31}
$2R_2+R_3$	$-2R_2+R_3$	L_{32}

ex. $A=\begin{bmatrix}-2 & 1 & 3\\ 4 & -2 & 1\\ 6 & 4 & 5\end{bmatrix}$, find LU factorization.

$$A=\begin{bmatrix}-2 & 1 & 3\\ 4 & -2 & 1\\ 6 & 4 & 5\end{bmatrix}\xrightarrow[3R_1+R_2]{2R_1+R_2}\begin{bmatrix}-2 & 1 & 3\\ 0 & 0 & 7\\ 0 & 7 & 14\end{bmatrix}\rightarrow\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

row op. $\underline{\underline{3}}$

بس مسموح

Not Possible.

ex= $A=\begin{bmatrix}0 & 2 & 3\\ -1 & 1 & 2\\ 1 & 2 & d\end{bmatrix}$, find LU factorization.

$$A=\begin{bmatrix}0 & 2 & 3\\ -1 & 1 & 2\\ 1 & 2 & d\end{bmatrix}\longrightarrow$$ not Possible...

ex. $A=\begin{bmatrix}-2 & 1 & 3\\ 4 & -2 & 1\\ 6 & -3 & 5\end{bmatrix}\xrightarrow[3R_1+R_3]{+2R_1+R_2}\begin{bmatrix}-2 & 1 & 3\\ 0 & 0 & 7\\ 0 & 0 & 14\end{bmatrix}\leftarrow U$

$$L=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}$$

عكس ال row op.

ما صفرته بوله أي شي فبنحط zero.

* ممكن يكون موجود أكثر من L, U .

to make sure :

$L.U = A$

باختصار
L U factorisation

السؤال بيجي على واحد منهم

إما بيعطيك L و U ، $Ax = b$ ، $A = LU$
وبحكيلك أوجدي x
من خلال
① نفرض $y = Ux$
② بحل معادلة $Ly = b$ ، بوجد y .
③ وأخيرا برجع بعوض في النقطة ① وبوجد x .

أو بعطيك A
وبحكيلك أوجد
U و L .
⇓
وهون فقط بستخدم row operation رقم 3 .

- ✦ وبوجد U .
- ✦ و L بتكون عبارة عنه

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$$

بكون عكس العمليات اللي عملناهم على U .