8.7 Part 1

15 February 2021 12:04

Improper Integrals

Type I

$$\int_a^{\infty} \quad \text{or} \quad \int_{-\infty}^{b} \quad \text{or} \quad \int_{-\infty}^{\infty}$$

$$\int_1^{\infty} \frac{dx}{x^2} \quad \text{I}$$

$$\int_2^{\infty} \frac{dx}{x-1} \quad \text{I}$$

$$\int_{-\infty}^{4} \frac{dx}{x^2} \quad \text{I} \quad \text{II}$$

$x=0$

Type II

$$\int_{\boxed{0}}^{\boxed{1}} \frac{1}{x}\, dx$$

$f(x)$

II

$$\int_0^2 \frac{dx}{x-1}$$

$x=1$?!

$$\boxed{\int_{-2}^{\infty} \frac{dx}{x+2}} \rightarrow \text{I} + \text{II}$$

$x=-2$

Question: How to find the

Question: How to find the Improper Integral ?!

Answer: Start with I

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f is cont. function on $[a, \infty) \Longrightarrow$

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx = \ldots$$

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f cont. on $(-\infty, a] \Longrightarrow$

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{b \to -\infty} \int_b^a f(x)\,dx$$

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f cont. on $(-\infty, \infty) \Longrightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\boxed{c}} f(x)\,dx + \int_{\boxed{c}}^{\infty} f(x)\,dx$$

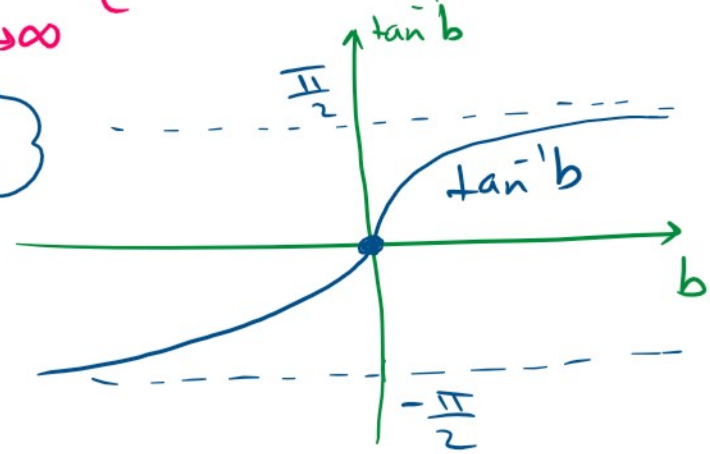
$\pm\infty$ $\qquad -\infty \qquad \boxed{c}$

$$= \lim_{b\to -\infty}\int_b^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx$$

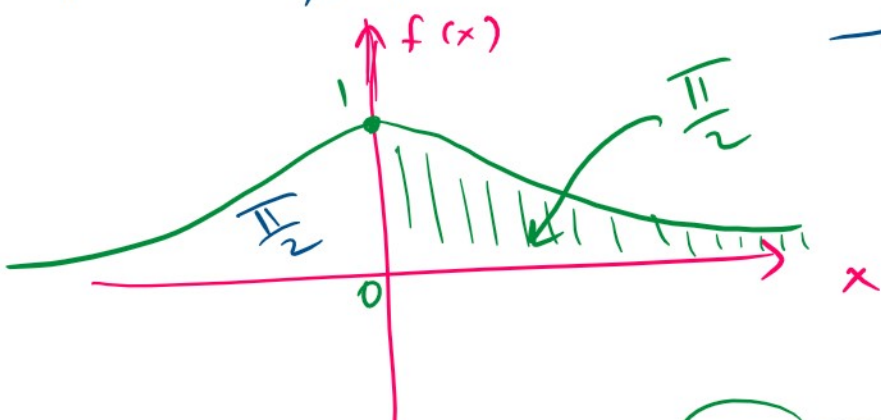
Exp $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1}$ $\quad$ I

$$= \lim_{b\to\infty} \tan^{-1}x \Big|_0^b = \lim_{b\to\infty}\left[\tan^{-1}b - \cancel{\tan^{-1}0}\right]$$

$$= \frac{\pi}{2}$$



$$f(x) = \frac{1}{x^2+1}$$



Exp $\displaystyle\int^{0} \frac{dx}{\quad} = \frac{\pi}{2}$

Exp $\displaystyle\int_{-\infty} \frac{dx}{x^2+1} = \left(\frac{1}{2} \right)$

Exp $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{\boxed{0}} \frac{dx}{x^2+1} + \int_{\boxed{0}}^{\infty} \frac{dx}{x^2+1}$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

Remark ① If the limit is finite (exists) then the Improper Integral converges

② If the limit DNE (or Infinite) then the Improper Integral diverges

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Exp $\displaystyle\int_{1}^{\infty} \frac{dx}{x} = \lim_{b \to \infty} \int_{1}^{b} \frac{dx}{x}$

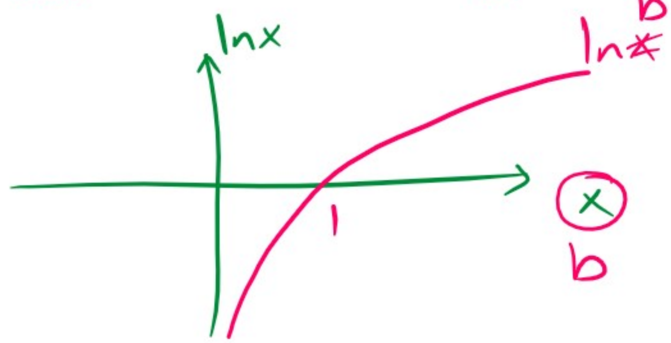
I

b

$$= \lim_{b\to\infty} \ln|x| \Big|_1^b = \lim_{b\to\infty} \left[\ln|b| - \underbrace{\ln|1|}_{0}\right]$$

$$= \boxed{\lim_{b\to\infty} \ln b}$$

$$= \underline{\underline{\infty}}$$



$$\int_1^\infty \frac{dx}{x} \text{ diverges}$$

$$\overbrace{\hspace{6cm}}^{\text{Improper Integral}}$$

I

$$\int_{-\infty}^{a}, \quad \int_{a}^{\infty}, \quad \int_{-\infty}^{\infty}$$

✓

II

$$\int_0^3 \frac{dx}{\sqrt{x}} \qquad f(x) = \frac{1}{\sqrt{x}}$$

Exp

I

$$\int_{-\infty}^{-2} \frac{2}{x^2-1}\, dx$$

converges to ln3

$$x^2 - 1 = 0$$

$$(x-1)(x+1) = 0$$

$$\notin (-\infty, -2]$$

$(x-1)(x+1) =$

$\boxed{x=1,\ x=-1} \notin (-\infty,-2]$

$(-\infty, -2]$

$$\lim_{b\to-\infty}\int_b^{-2} \frac{2\,dx}{x^2-1} =$$

$$\frac{2}{x^2-1} = \frac{2}{\cancel{(x-1)}\cancel{(x+1)}} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$A = \frac{2}{(1)+1} = 1$$

$$B = \frac{2}{(-1)-1} = -1$$

$$\lim_{b\to-\infty}\int_b^{-2}\left(\frac{A}{x-1} + \frac{B}{x+1}\right)dx$$

$$\lim_{b\to-\infty}\int_b^{-2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx$$

$$\lim_{b\to-\infty}\Big[\ln|x-1| - \ln|x+1|\Big]\Big|_b^{-2}$$

$$\lim \ \ln\left|\frac{x-1}{\ldots}\right| \Big|^{-2}$$

$$\lim_{b \to -\infty} \ln\left|\frac{x-1}{x+1}\right| \Bigg|_b$$

$$\lim_{b \to -\infty} \left[\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right| \right]$$

cont.

$$\ln 3 - \lim_{b \to -\infty} \ln\left|\frac{b-1}{b+1}\right|$$

$$\ln 3 - \ln\left(\lim_{b \to -\infty} \frac{b-1}{b+1} \right) \qquad \frac{-\infty}{-\infty} \quad \frac{\infty}{\infty}$$

$$\ln 3 - \ln 1 \to 0$$

$$\ln 3$$ finint

Exp $\displaystyle\int_1^\infty \frac{dx}{x^1}$ diverges (∞) by Exp*

Exp* I

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1 \\ \infty & p \le 1 \end{cases}$$

Exp** II

$$\int_0^1 \frac{dx}{x^p} = \begin{cases} \dfrac{1}{1-p} & \text{if } p<1 \\ \infty & \text{if } p \geq 1 \end{cases}$$

Exp ① $\displaystyle\int_1^\infty \frac{dx}{x^3} = \frac{1}{3-1} = \frac{1}{2}$ since $p=3>1$

I converges by Exp*

② $\displaystyle\int_1^\infty \frac{dx}{x^{2/3}} = \infty$ since $p = \frac{2}{3} \leq 1$

③ $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}} = \infty$ since $p = \frac{1}{2}$ by Exp*

④ $\displaystyle\int^1 \frac{dx}{} = \frac{1}{} = \frac{1}{} = 2$

(4) $\int_0 \frac{dx}{\sqrt{x}} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

since $p = \frac{1}{2}$ by Exp **

Exp

Type II

$$\int_0^4 \frac{dx}{\sqrt{4-x}}$$ converges to 4

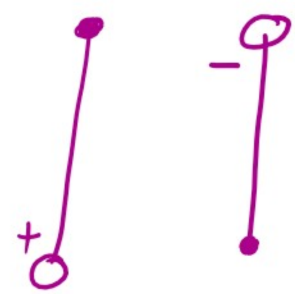
not I

$\boxed{x = 4}$

$$\lim_{c \to 4^-} \int_0^c \frac{2\,dx}{2\sqrt{4-x}}$$

$u = 4 - x$

$du = -dx$



$$\lim_{c \to 4^-} (-2)\sqrt{4-x}\,\Big|_0^c$$

$$\lim_{c \to 4^-} -2\left[\sqrt{4-c} - \sqrt{4-0}\right]$$

$$\lim_{c \to 4^-} \left[-2\sqrt{4-c} + (2)(2)\right]$$

$$-2\sqrt{4-4} + 4 = 4$$

$-2\sqrt{4-4} + 4 = 4$

Exp $\int_0^1 \frac{dx}{\sqrt{x}}$ II

$x=0$

$\frac{1}{2}\sqrt{x}$

$\lim_{c\to 0^+} 2\int_c^1 \frac{dx}{2\sqrt{x}} = \lim_{c\to 0^+} 2\sqrt{x}\Big|_c^1 = \lim_{c\to 0^+}\left(2\sqrt{1}-2\sqrt{c}\right)$

$= 2$

converges to $\sqrt{3}$

Exp 15 $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

not I

II ??

$\theta^2+2\theta = \theta(\theta+2)$

$\theta = 0$ $\quad$ $\cancel{\theta=-2}$

$\lim_{c\to 0^+}\int_c^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

$u = \theta^2+2\theta$

$du = (2\theta+2)\,d\theta$

$\frac{du}{2} = (\theta+1)\,d\theta$

$\lim_{c\to 0^+}\int_c^1 \frac{\frac{du}{2}}{\sqrt{u}}$

$$\lim_{c\to 0^+} \int_c^1 \frac{du}{2\sqrt{u}} = \lim_{c\to 0^+} \sqrt{u}\Big|_c^1$$

$$= \lim_{c\to 0^+} \sqrt{\theta^2+2\theta}\Big|_c^1$$

$$= \lim_{c\to 0^+} \underline{\underline{\sqrt{1+2}}} - \sqrt{\underset{\uparrow 0}{c^2}+\underset{\uparrow 0}{2c}}$$

$$\sqrt{3} - 0$$

$$\sqrt{3}$$

Exp $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$ Converges to $\frac{\pi}{2}$

I ✓

II ✓

$[1, \infty)$

$$\underbrace{\int_1^2 \frac{dx}{x\sqrt{x^2-1}}}_{\text{II}} + \underbrace{\int_2^\infty \frac{dx}{x\sqrt{x^2-1}}}_{\text{I}}$$

± 1, 0

$[1 \quad 2 \quad \infty$

$\int^b \; dx$

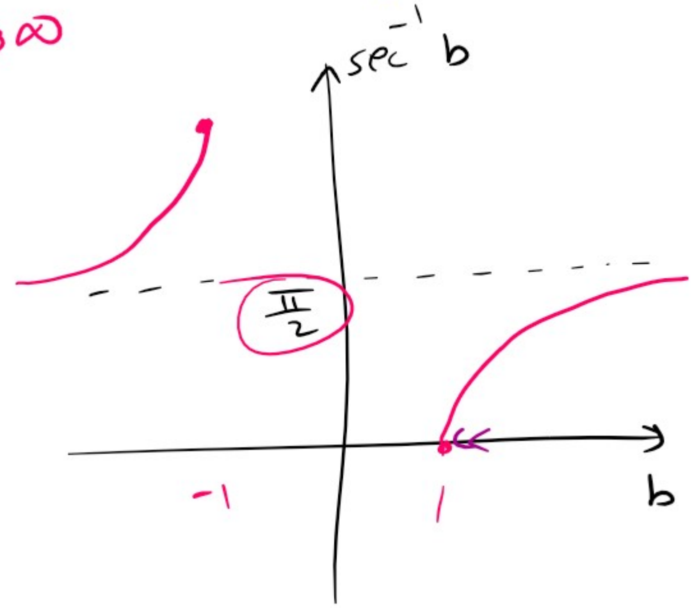
$1 \qquad 2$

$$\lim_{c\to 1^+}\int_c^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{b\to\infty}\int_2^b \frac{dx}{x\sqrt{x^2-1}}$$

$$\lim_{c\to 1^+} \sec^{-1}|x|\Big|_c^2 + \lim_{b\to\infty} \sec^{-1}|x|\Big|_2^b$$

$$\lim_{c\to 1^+} \cancel{\sec^{-1} 2}^{\frac{\pi}{3}} - \sec^{-1} c + \lim_{b\to\infty} \sec^{-1} b - \cancel{\sec^{-1} 2}^{\frac{\pi}{3}}$$

$$\sec^{-1} 2 = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$



$$\lim_{c\to 1^+} \sec^{-1} c + \lim_{b\to\infty} \sec^{-1} b$$

$$0 + \frac{\pi}{2}$$

$$\frac{\pi}{2}$$

Exp $\displaystyle\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx$

I

$$\lim \int^b \frac{16\tan^{-1}x}{\ldots}\,dx \qquad u = \tan^{-1}x$$

$16\,dx$

$$\lim_{b\to\infty}\int_0^b \frac{16\tan^{-1}x}{1+x^2}\,dx$$

$$16\,du = \frac{16\,dx}{1+x^2}$$

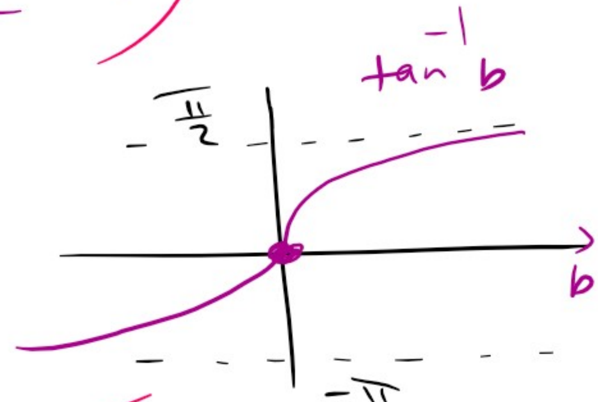
$$\lim_{b\to\infty}\int_0^b 16\,u\,du = \lim_{b\to\infty} 8u^2\Big|_0^b$$

$$= \lim_{b\to\infty} 8\left(\tan^{-1}x\right)^2\Big|_0^b$$

$$= 8\lim_{b\to\infty}\left(\left(\tan^{-1}b\right)^2 - \left(\tan^{-1}0\right)^2\right)$$

$$= 8\left(\left(\frac{\pi}{2}\right)^2 - 0\right)$$

$$= \frac{\pi^2}{4}\,8 = 2\pi^2$$



$$a_n = \frac{1}{n}$$

converges

$$a_1 = \frac{1}{1} = 1$$

$$a_2 = \frac{1}{2}$$

$$a_3 = \frac{1}{3}$$

Converges to zero

$a_3 = \frac{1}{3}$

$\vdots$

$a_{100} = \frac{1}{100}$

$\vdots$

$a_\infty = 0$