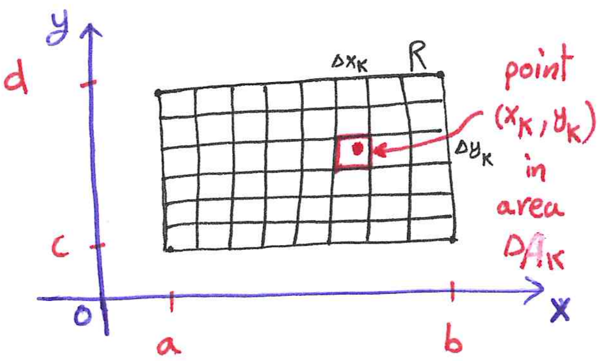
15.1 Double and Itereated Integrals over Rectangles

How to construct double integral ?

- Assume $f(x,y)$ is defined on a rectangular region R

$$R: \ a \le x \le b \ , \ c \le y \le d$$

- Divide R into n small rectangulars with width Δx and height Δy



- Each small rectangle has area

$$\Delta A = \Delta x \, \Delta y$$

- These n small rectangles form a partition P and the number n gets large as Δx and Δy become smaller.

- If we order the areas $\Delta A_1, \Delta A_2, \dots, \Delta A_K, \dots, \Delta A_n$ and in each ΔA_K "small rectangle" we choose a point (x_K, y_K) and evaluate $f(x_K, y_K)$ "hieght", then the Riemann sum over R is

$$S_n = \sum_{K=1}^{n} f(x_K, y_K) \, \Delta A_K$$

- As $\Delta x \to 0$ and $\Delta y \to 0$ the norm of the partition $\|P\| \to 0$ Hence, $n \to \infty$, where $\|P\| = \max\{\Delta x, \Delta y\}$ for any rectangle.

- Therefore, $\lim\limits_{\|P\| \to 0} S_n = \lim\limits_{\|P\| \to 0} \sum_{K=1}^{n} f(x_K, y_K) \Delta A_K = \lim\limits_{n \to \infty} \sum_{K=1}^{n} f(x_K, y_K) \Delta A_K$

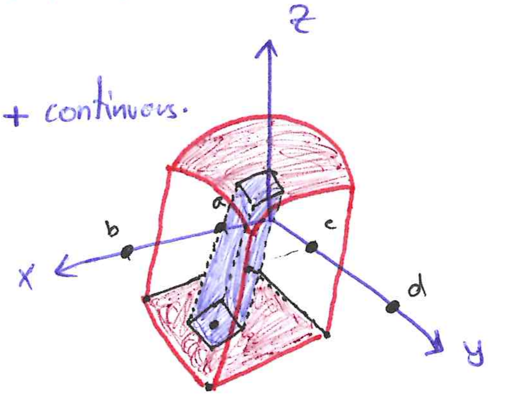
- If the limit exists, then its called double Integral :

$$\iint_R f(x,y) \, dA \quad \text{or} \quad \iint_R f(x,y) \, dx \, dy$$

and the function f is said to be integrable.

* The volume of the resulting solid is

$$V = \iint_R f(x,y)\, dA$$ if $f(x,y)$ is + continuous.



* Fubini's Theorem for Calculating Double Integrals :

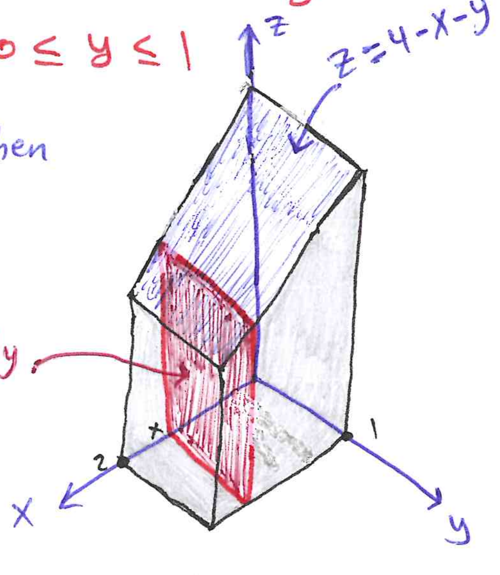
Exp Find the volume under the plane $z = 4 - x - y$ over the region $R: 0 \le x \le 2$, $0 \le y \le 1$

• If cross section ⊥ x-axis is taken, then

The volume is

$$V = \int_{x=0}^{x=2} A(x)\, dx \quad \text{where} \quad A(x) = \int_{y=0}^{y=1} (4-x-y)\, dy$$

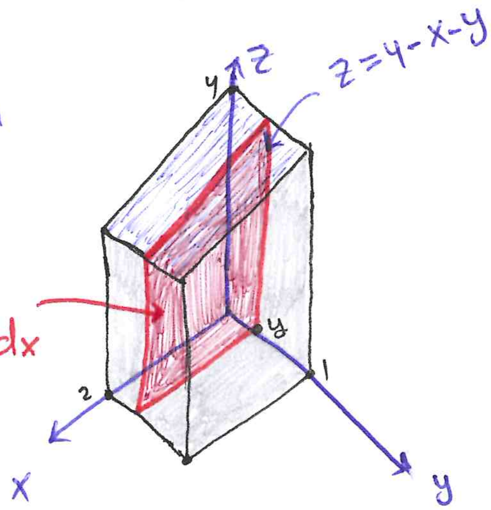
$$= \int_0^2 \int_0^1 (4-x-y)\, dy\, dx = 5$$



• If cross section ⊥ y-axis is taken, then the volume is

$$V = \int_{y=0}^{y=1} A(y)\, dy \quad \text{where} \quad A(y) = \int_{x=0}^{x=2} (4-x-y)\, dx$$

$$= \int_0^1 \int_0^2 (4-x-y)\, dx\, dy = 5$$



Th (Fubini's Theorem - First form)

If $f(x,y)$ is continous on rectangular region $R: a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Exp: Find the volume of the region bounded above by the plane $z = 2-x-y$ and below by the square $R: 0 \le x \le 1,\ 0 \le y \le 1$.

$$V = \int_0^1 \int_0^1 (2-x-y)\,dx\,dy = \int_0^1 \left(2x - \frac{x^2}{2} - yx\right)\Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{3}{2} - y\right) dy = \frac{3}{2}y - \frac{y^2}{2}\Big|_0^1 = 1$$

Exp Find the volume of the region bounded above by the surface $z = 2\sin x \cos y$ and below by the rectangle $R: 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{4}$.

$$V = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} 2\sin x \cos y\,dy\,dx = \int_0^{\frac{\pi}{2}} \left(2\sin x \sin y\Big|_{y=0}^{y=\frac{\pi}{4}}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{2}\sin x\,dx = -\sqrt{2}\cos x\Big|_0^{\frac{\pi}{2}} = \sqrt{2}$$
