4.3 The Method of Undetermined Coefficients

A particular solution Y of the nonhomogeneous nth order linear equation with constant coefficients

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \tag{1}$$

can be obtained by the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form. Although the method of undetermined coefficients is not as general as the method of variation of parameters described in the next section, it is usually much easier to use when it is applicable.

Method of Undetermined Coefficients

To find a particular solution to the constant-coefficient differential equation $L[y] = Cx^m e^{rx}$, where m is a nonnegative integer, use the form

$$y_p(x) = x^s[A_m x^m + \cdots + A_1 x + A_0]e^{rx},$$

with $s = 0$ if r is not a root of the associated auxiliary equation; otherwise, take s equal to the multiplicity of this root.

To find a particular solution to the constant-coefficient differential equation $L[y] = Cx^m e^{\alpha x}\cos\beta x$ or $L[y] = Cx^m e^{\alpha x}\sin\beta x$, where $\beta \neq 0$, use the form

$$y_p(x) = x^s[A_m x^m + \cdots + A_1 x + A_0]e^{\alpha x}\cos\beta x + x^s[B_m x^m + \cdots + B_1 x + B_0]e^{\alpha x}\sin\beta x,$$

with $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; otherwise, take s equal to the multiplicity of this root.

EXAMPLE 1

Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t. \tag{2}$$

The characteristic polynomial for the homogeneous equation corresponding to Eq. (2) is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3,$$

so the general solution of the homogeneous equation is

$$y_c(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t. \tag{3}$$

To find a particular solution $Y(t)$ of Eq. (2), we start by assuming that $Y(t) = Ae^t$. However, since e^t, te^t, and t^2e^t are all solutions of the homogeneous equation, we must multiply this initial choice by t^3. Thus our final assumption is that $Y(t) = At^3e^t$, where A is an undetermined coefficient. To find the correct value for A, we differentiate $Y(t)$ three times, substitute for y and its derivatives in Eq. (2), and collect terms in the resulting equation. In this way we obtain

$$6Ae^t = 4e^t.$$

Thus $A = \frac{2}{3}$ and the particular solution is

$$Y(t) = \tfrac{2}{3}t^3e^t. \tag{4}$$

The general solution of Eq. (2) is the sum of $y_c(t)$ from Eq. (3) and $Y(t)$ from Eq. (4):

$$y = c_1e^t + c_2te^t + c_3t^2e^t + \tfrac{2}{3}t^3e^t.$$

EXAMPLE 2

Find a particular solution of the equation

$$y^{(4)} + 2y'' + y = 3\sin t - 5\cos t. \qquad (5)$$

The general solution of the homogeneous equation was found in Example 3 of Section 4.2; it is

$$y_c(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t, \tag{6}$$

corresponding to the roots $r = i, i, -i$, and $-i$ of the characteristic equation. Our initial assumption for a particular solution is $Y(t) = A \sin t + B \cos t$, but we must multiply this choice by t^2 to make it different from all solutions of the homogeneous equation. Thus our final assumption is

$$Y(t) = At^2 \sin t + Bt^2 \cos t.$$

Next, we differentiate $Y(t)$ four times, substitute into the differential equation (4), and collect terms, obtaining finally

$$-8A \sin t - 8B \cos t = 3 \sin t - 5 \cos t.$$

Thus $A = -\frac{3}{8}$, $B = \frac{5}{8}$, and the particular solution of Eq. (4) is

$$Y(t) = -\frac{3}{8}t^2 \sin t + \frac{5}{8}t^2 \cos t. \tag{7}$$

EXAMPLE 3

Find a particular solution of

$$y''' - 4y' = t + 3\cos t + e^{-2t}. \tag{8}$$

First we solve the homogeneous equation. The characteristic equation is $r^3 - 4r = 0$, and the roots are $r = 0, \pm 2$; hence

$$y_c(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}.$$

We can write a particular solution of Eq. (8) as the sum of particular solutions of the differential equations

$$y''' - 4y' = t, \qquad y''' - 4y' = 3\cos t, \qquad y''' - 4y' = e^{-2t}.$$

Our initial choice for a particular solution $Y_1(t)$ of the first equation is $A_0 t + A_1$, but a constant is a solution of the homogeneous equation, so we multiply by t. Thus

$$Y_1(t) = t(A_0 t + A_1).$$

For the second equation we choose

$$Y_2(t) = B\cos t + C\sin t,$$

and there is no need to modify this initial choice since $\sin t$ and $\cos t$ are not solutions of the homogeneous equation. Finally, for the third equation, since e^{-2t} is a solution of the homogeneous equation, we assume that

$$Y_3(t) = Ete^{-2t}.$$

The constants are determined by substituting into the individual differential equations; they are $A_0 = -\frac{1}{8}$, $A_1 = 0$, $B = 0$, $C = -\frac{3}{5}$, and $E = \frac{1}{8}$. Hence a particular solution of Eq. (8) is

$$Y(t) = -\tfrac{1}{8}t^2 - \tfrac{3}{5}\sin t + \tfrac{1}{8}te^{-2t}. \tag{9}$$

In each of Problems 1 through 8, determine the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 2e^{-t} + 3$
2. $y^{(4)} - y = 3t + \cos t$
3. $y''' + y'' + y' + y = e^{-t} + 4t$
4. $y''' - y' = 2\sin t$
5. $y^{(4)} - 4y'' = t^2 + e^t$
6. $y^{(4)} + 2y'' + y = 3 + \cos 2t$
7. $y^{(6)} + y''' = t$
8. $y^{(4)} + y''' = \sin 2t$

7. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r^3+1)=0$. Thus the homogeneous solution is

$$y_c = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + e^{t/2}\left[c_5 cos\left(\sqrt{3}\, t/2\right) + c_5 sin\left(\sqrt{3}\, t/2\right)\right].$$

Note the $g(t)=t$ is a solution of the homogenous problem. Consider a particular solution
of the form $Y(t)=t^3(At+B)$. Substitution into the ODE results in $A=1/24$ and $B=0$. Thus the general solution is $y(t)=y_c(t)+t^4/24$.

In each of Problems 13 through 18, determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

13. $y''' - 2y'' + y' = t^3 + 2e^t$
14. $y''' - y' = te^{-t} + 2\cos t$
15. $y^{(4)} - 2y'' + y = e^t + \sin t$
16. $y^{(4)} + 4y'' = \sin 2t + te^t + 4$
17. $y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t$
18. $y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$

18. The characteristic equation can be written as $r^2(r^2+2r+2)=0$, with roots $r=0$, with multiplicity *two*, and $r=-1\pm i$. The homogeneous solution is $y_c = c_1 + c_2 t + c_3 e^{-t} cos\, t + c_4 e^{-t} sin\, t$. The function $g_1(t) = 3e^t + 2te^{-t}$, and all of its derivatives, is independent of the homogeneous solution. Therefore set $Y_1(t) = Ae^t + (Bt+C)e^{-t}$. Now $g_2(t) = e^{-t} sin\, t$ is a solution of the homogeneous equation, associated with the complex roots. We need to set $Y_2(t) = t(D e^{-t} cos\, t + E e^{-t} sin\, t)$. It follows that the particular solution has the form

$$Y(t) = Ae^t + (Bt+C)e^{-t} + t\left(D e^{-t} cos\, t + E e^{-t} sin\, t\right).$$

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

9. $y''' + 4y' = t; \quad y(0) = y'(0) = 0, \quad y''(0) = 1$
10. $y^{(4)} + 2y'' + y = 3t + 4; \quad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1$
11. $y''' - 3y'' + 2y' = t + e^t; \quad y(0) = 1, \quad y'(0) = -\frac{1}{4}, \quad y''(0) = -\frac{3}{2}$
12. $y^{(4)} + 2y''' + y'' + 8y' - 12y = 12\sin t - e^{-t}; \quad y(0) = 3, \quad y'(0) = 0,$
    $y''(0) = -1, \quad y'''(0) = 2$

11. The characteristic equation can be written as $r(r^2 - 3r + 2) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2 e^t + c_3 e^{2t}$. Let $g_1(t) = e^t$ and $g_2(t) = t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1(t) = Ate^t$. Substitution into the ODE results in $A = -1$. Now let $Y_2(t) = Bt^2 + Ct$. Substitution into the ODE results in $B = 1/4$ and $C = 3/4$. Therefore the general solution is

$$y(t) = c_1 + c_2 e^t + c_3 e^{2t} - te^t + (t^2 + 3t)/4 .$$

Invoking the initial conditions, we find that $c_1 = 1$, $c_2 = c_3 = 0$. The solution of the initial value problem is $y(t) = 1 - te^t + (t^2 + 3t)/4$.