Chapter 3 (3.1 + 3.2)

1. $\lim_{x \to a} f(x) = L$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

2. $\lim_{x \to a^+} f(x) = L$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t $a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$.

3. $\lim_{x \to a^-} f(x) = L$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t $a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon$.

4. $\lim_{x \to \infty} f(x) = L$ iff $\varepsilon > 0$, $\exists M \in \mathbb{R}$ s.t $x > M \Rightarrow |f(x) - L| < \varepsilon$.

5. $\lim_{x \to -\infty} f(x) = L$ iff $\varepsilon > 0$, $\exists M \in \mathbb{R}$ s.t $x < M \Rightarrow |f(x) - L| < \varepsilon$.

6. $\lim_{x \to a} f(x) = \infty$ iff $M \in \mathbb{R}$, $\exists \delta > 0$ s.t $0 < |x-a| < \delta \Rightarrow f(x) > M$.

7. $\lim_{x \to a} f(x) = -\infty$ iff $M \in \mathbb{R}$, $\exists \delta > 0$ s.t $0 < |x-a| < \delta \Rightarrow f(x) < M$.

8. $\lim_{x \to a^+} f(x) = \infty$ iff $M \in \mathbb{R}$, $\exists \delta > 0$ s.t $a < x < a + \delta \Rightarrow f(x) > M$.

9. $\lim_{x \to a^-} f(x) = \infty$ iff $M \in \mathbb{R}$, $\exists \delta > 0$ s.t $a - \delta < x < a \Rightarrow f(x) > M$.

8,9 : same since $f(x) \to -\infty$ but change $f(x) < M$.

3.3: Continuity

Def 1: let $\emptyset \neq E \subseteq \mathbb{R}$ and $f: E \to \mathbb{R}$

i. f is said to be continuous at a point $a \in E$ iff $\forall \varepsilon > 0$, $\exists\, \delta > 0$ (depends on ε, f, a) s.t $|x-a| < \delta$ and $x \in E \Rightarrow |f(x) - f(a)| < \varepsilon$.

ii. f is said to be continuous on E iff f is continuous at every $x \in E$.

RMK: let I be an open interval which contains a point a and $f: I \to \mathbb{R}$ Them f is continuous at $a \in I$ iff $f(a) = \lim_{x \to a} f(x)$.

Thm 1: sequential caracterization of continuity:

suppose that E is a nonempty subset of $\mathbb{R}$, that $a \in E$ and that $f: E \to \mathbb{R}$ Then the following statments are equivalent.

i. f is continuous at $a \in E$.

ii. If $x_n \to a$ and $x_n \in E$ then $f(x_n) \to f(a)$ as $n \to \infty$.

Thm 2: let E be anonempty subset of $\mathbb{R}$ and $f, g: E \to \mathbb{R}$. If f, g are continuous at a point $a \in E$ (resp. continuous on the set E), then so are $f+g$, fg and αf ($\alpha \in \mathbb{R}$) Moreover $\frac{f}{g}$ is continuous at $a \in E$ when $g(a) \neq 0$ (resp. on E when $g(x) \neq 0\ \forall x \in E$).

Def 2: suppose that A and B are subset of $\mathbb{R}$, that $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$

If $f(a) \subseteq B$ for every $x \in A$ then the composition of g with f is the function $g \circ f: A \to \mathbb{R}$ defind By $(g \circ f)(x) := g(f(x))$, $x \in A$.

Thm 3: suppose that A and B are subset of $\mathbb{R}$, that $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and that $f(x) \in B$, $\forall x \in A$:

i. If $A := I \setminus \{a\}$ where I is a nondegenerate interval which either contains a or has a as one of its endpoints if $L := \lim_{\substack{x \to a \\ x \in I}} f(x)$ exists and Belong to B, and if g is cont. at $L \in B$ then

$$\lim_{\substack{x \to a \\ x \in I}} (g \circ f)(x) = g\left(\lim_{\substack{x \to a \\ x \in I}} f(x) \right).$$

ii. If f is cont. at $a \in A$ and g is cont. at $f(a) \in B$, then $g \circ f$ is cont. at $a \in A$.

Def 3: let $\emptyset \neq E \subseteq \mathbb{R}$, a function $f: E \to \mathbb{R}$ is said to be bounded on E iff $\exists$ an $M \in \mathbb{R}$ s.t $|f(x)| \leq M$, $\forall x \in E$. (f is dominated By M on E).

RMK: Notice that whether a function f is bounded or not on a set E depends on E as well as on f.

Thm 4: If I is closed, bounded interval and $f: I \to \mathbb{R}$ is continuous on I, then f is bounded on I. Moreover if $M = \sup_{x \in I} f(x)$ and $m \in \inf_{x \in I} f(x)$, then $\exists$ points $x_m, x_M \in I$ s.t $f(x_m) = m$ and $f(x_M) = M$. (Extreme value Thm).

RMK:

1. We also call the value M (resp. m) the Maximum (resp. Minimum) of f on I.
2. Extreme value Thm is false if either closed or bounded is dropped from the hypothesis.

Thm 5: Intermediate value Theorem.

suppose that $a < b$ and that $f: [a,b] \to \mathbb{R}$ is continuous If y_0 lies between $f(a)$ and $f(b)$ then $\exists$ an $x_0 \in (a,b)$ s.t $f(x_0) = y_0$.

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RMK : The composition of two function $g \circ f$ can be Nowhere continuous even though f is discontinuous ONLY on $\mathbb{Q}$ and g is discont. at only one point.

$$f(x) = \begin{cases} \frac{1}{q}, & x \in \mathbb{Q},\ x = \frac{p}{q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

$$g(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

3.4: uniform continuity :

Def 1 : let E be a nonempty subset of $\mathbb{R}$ and $f: E \to \mathbb{R}$, Then f is said to be uniformly continuous on E iff $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ s.t $|x-a| < \delta$ and $x, a \in E$ implies $|f(x) - f(a)| < \varepsilon$. (δ depends on ε and f, But Not on a and x).

RMK :

1. الفرق بين continuous و uniform conti.
2. Every uniformly continuous function on E is also continuous on E. But the converse not true.

Non uniformly continuity criteria :

let $E \subseteq \mathbb{R}$ and let $f: E \to \mathbb{R}$ Then the following statements are equivalent :

i. f is Not uniformly continuous on E.

ii. $\exists$ an $\varepsilon_0 > 0$ s.t $\forall \delta > 0$ there are points $x, y \in E$ s.t $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$.

ii. $\exists$ an $\varepsilon_0 > 0$ and two sequences $x_n, y_n \in E$ s.t $x_n - y_n \to 0$ as $n \to \infty$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$, $\forall n \in \mathbb{N}$.

lemma : suppose that $E \subseteq \mathbb{R}$ and that $f: E \to \mathbb{R}$ is uniformly continuous, If $x_n \in E$ is cauchy then $\{f(x_n)\}$ is cauchy.

Thm 1 : suppose that I is closed, bounded interval, If $f: I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

Thm 2 : suppose that $a < b$ and that $f: (a,b) \to \mathbb{R}$ Then f is uniformly continuous on (a,b) iff f can be continuously extended to $[a,b]$, i.e iff there is a continuous function $g: [a,b] \to \mathbb{R}$ which satisfies $f(x) = g(x)$, $x \in (a,b)$.

RMK : let f be cont. on a bounded, open, nondegenerate interval (a,b), Notice that f is conti. extendable to $[a,b]$ iff $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist. Indeed where they exist we define g at $x=a$ and $x=b$ as $g(a) = \lim_{x \to a^+} f(x)$, $g(b) = \lim_{x \to b^-} f(x)$.