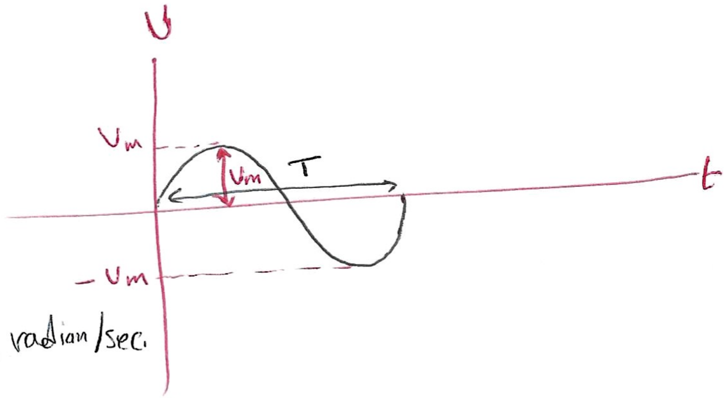
Chapter 9 : Sinusoidal steady - state Analysis

9.1 : the Sinusoidal source



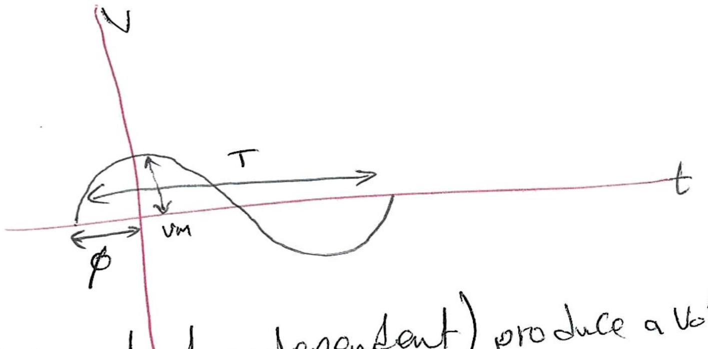
$$V_s(t) = V_m \sin(wt)$$

T : period in seconds

$f = \frac{1}{T}$ frequency in HZ

$w = 2\pi f$ angular frequency in radian/sec.

$$V_s(t) = V_m \cos(wt + \phi)$$



A Sinusoidal voltage/current source :- (independent or dependent) produce a voltage, current that varies sinusoidally with time.

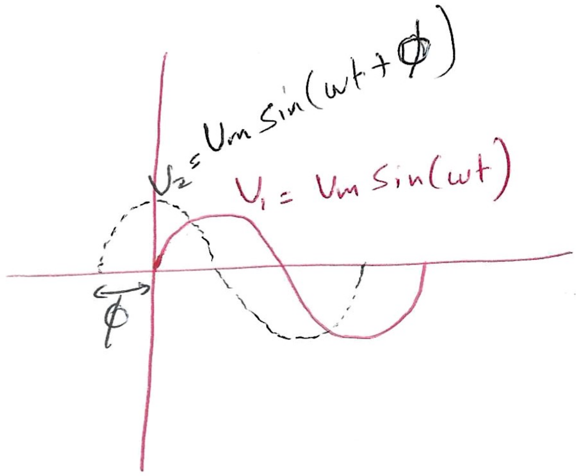
V_{rms} Root Mean square Value

$$V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(wt+\phi)\,dt} = \frac{V_m}{\sqrt{2}}$$

V_2 Leads V_1 by phase ϕ

V_1 lags V_2 by phase ϕ

V_1 and V_2 are out of phase



Example:- let $V_1(t) = 10\sin(5t - 30°)$

$V_2(t) = 15\sin(5t + 10°)$

∴ V_2 leads V_1 by $40°$

V_1 lags V_2 by $40°$

Example:- let $i_1(t) = \overset{I_{m1}}{2}\sin(\overset{\omega_1}{377}t + \overset{\phi_1}{45°})$

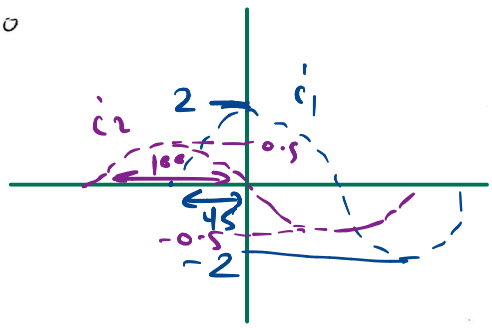
$i_2(t) = \overset{I_m}{0.5}\cos(\overset{\omega_2}{377}t + \overset{\phi_2}{10°})$

$-\sin\alpha = \cos(\alpha + 90°)$

$\cos\alpha = \sin(\alpha + 90°)$

$0.5\cos(377t + 10°) = 0.5\sin(377t + 100°)$

∴ i_2 leads $i_1(t)$ by $55°$



Trigonometric Identities

$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$

$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$

$\sin(wt \pm 180°) = -\sin(wt)$

$\cos(wt \pm 180) = -\cos(wt)$

$\sin(wt \pm 90°) = \pm\cos(wt)$

$\cos(wt \pm 90) = \mp\sin(wt)$

$A\cos(wt) + B\sin(wt) = C\cos(wt - \theta)$

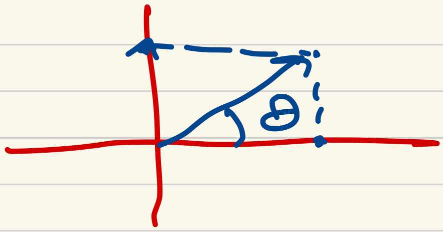
$C = \sqrt{A^2 + B^2}$ $\qquad \theta = \tan^{-1}\left(\frac{B}{A}\right)$

* Complex numbers :

- → Rectangular / cartesian $z = a + jb$
- → exponential $|z| e^{j\theta}$
- → phasar / polar $|z| \angle\theta$

$a = |z| \cos\theta$

$b = |z| \sin\theta$



$$z = a + jb \Rightarrow |z| = \sqrt{a^2 + b^2}$$

$$\angle\theta = \tan^{-1} \frac{b}{a}$$

* Euler

$$A e^{j\theta} = A(\cos\theta + j\sin\theta)$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Example: A Sinusoidal current has a maximum amplitude of 20A.

The current passes through one complete cycle in 1ms.

The magnitude of the current at zero time is 10A.

$I_m = 20A$

$T = 1ms$

$I(0) = 10A$

a) what is the frequency of the current in hertz?

$T = 1ms \Rightarrow f = \frac{1}{T} = 1000\ Hz$

b) what is the frequency in radians per second?

$w = 2\pi f = 2\pi(1000) = 2000\pi$ rad/s

c) write the expression for $i(t)$ using the cosine function. Express ϕ in degree.

$i(t) = I_m \cos(wt + \phi)$

$= 20 \cos(2000\pi t + \phi)$

but at $t = 0$ $i(0) = 10A$

$i(0) = 10A = 20\cos(\phi)$

$\Rightarrow \phi = 60°$

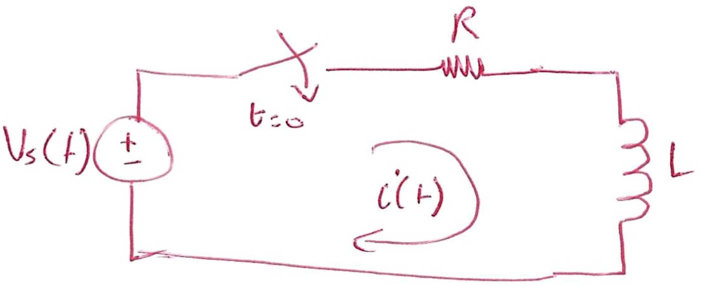
$\therefore i(t) = 20\cos(2000\pi t + 60°)$ A

d) what is the rms value of the current?

$I_{rms} = \frac{I_m}{\sqrt{2}} = \frac{20}{\sqrt{2}} = 14.14\ A$

9.2 The Sinusoidal Response

Find $i(t)$ for $t > 0$



given $V_s(t) = V_m \cos(\omega t)$ V

given that $i_L(0^-) = 0$

KVL

$$V_s(t) = R\,i(t) + L\frac{di(t)}{dt}$$

$$V_m \cos(\omega t) = R\,i(t) + L\frac{di(t)}{dt}$$ (first order non homogenouse differential equation)

$$\therefore i(t) = i_{in}(t) + i_f(t)$$

in : initial
f : final

$$= A e^{-t/\tau} + i_f(t)$$

$$i_f(t) = I_1 \cos(\omega t) + I_2 \sin(\omega t)$$

to find I_1 and I_2

$$V_m \cos(\omega t) = R\,i(t) + L\frac{di}{dt}$$

$$V_m \cos(\omega t) = R\left[I_1 \cos(\omega t) + I_2 \sin(\omega t)\right] + L\omega\left[-I_1 \sin(\omega t) + I_2 \cos(\omega t)\right]$$

$$= (-L I_1 \omega + R I_2)\sin(\omega t) + (L I_2 \omega + R I_1)\cos(\omega t)$$

$$\therefore \quad -\omega L I_1 + R I_2 = 0 \quad \text{--- (1)}$$

$$\omega L I_2 + R I_1 = V_m \quad \text{--- (2)}$$

$$I_1 = \frac{R V_m}{R^2 + \omega^2 L^2}$$

$$I_2 = \frac{\omega L V_m}{R^2 + \omega^2 L^2}$$

$$\therefore i_f(t) = \frac{R V_m}{R^2 + \omega^2 L^2}\cos(\omega t) + \frac{\omega L V_m}{R^2 + \omega^2 L^2}\sin(\omega t)$$

from the identity $A\cos(\omega t) + B\sin(\omega t)$

$$i_f(t) = C\cos(\omega t - \phi)$$

$$i_f(t) = C\cos(\omega t)\cos(\phi) + C\sin(\omega t)\sin(\phi)$$

$I_1 = C\cos(\phi)$

$I_2 = C\sin(\phi)$

$$\frac{I_2}{I_1} = \frac{C\sin(\phi)}{C\cos(\phi)} = \tan\phi$$

$$\phi = \tan^{-1}\left(\frac{I_2}{I_1}\right) = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$I_1^2 + I_2^2 = C^2\cos^2\phi + C^2\sin^2\phi$$
$$= C^2(1)$$
$$= C^2$$

$$\therefore C = \sqrt{I_1^2 + I_2^2}$$

$$C = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}$$

$$\therefore i_f(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

$$\therefore i(t) = i_n(t) + i_f(t)$$
$$= Ae^{-t/\tau} + \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

$i(0) = 0$

$$\therefore 0 = A + \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos\left(-\tan\left(\frac{\omega L}{R}\right)\right)$$

$$\therefore A = \frac{-V_m}{\sqrt{R^2+\omega^2L^2}}\cos\left(-\tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

$i(t)$ = transient component + steady state component the steady state solution

$$i_{ss}(t) = \frac{V_m}{\sqrt{R^2+w^2L^2}} \cos\left(wt - \tan^{-1}\left(\frac{wL}{R}\right)\right)$$

* the steady state solution is sinusoidal function
* the frequency of the solution is the same as the frequency of the source
* the maximum amplitude of the solution defers from the maximum amplitude of the source

  V_m for the source and $\frac{V_m}{\sqrt{R^2+w^2L^2}}$ for the solution
* the phase angle of the solution defers from the phase angle of the source.

9.3 The phasor

The phasor is a complex number carries the amplitude and phase angle information of a sinusoidal function.

$$e^{\pm j\theta} = \cos(\theta) \pm j\sin(\theta)$$

$$\cos(\theta) = Re\{e^{j\theta}\} \qquad \sin\theta = Im\{e^{j\theta}\}$$

$$\cos(wt+\theta) = Re\{e^{j(wt+\theta)}\}$$

$$V_m \cos(wt+\theta) = V_m Re\{e^{j(wt+\theta)}\} = Re\{V_m e^{jwt} e^{j\theta}\}$$

So phasor transform of $P(V_m \cos(\omega t + \theta)) = V_m e^{j\theta} = \vec{V}$

$$\vec{V} = V_m e^{j\theta} = V_m \angle \theta$$

Example: $i(t) = 6 \cos(50t - 40°)$ A

$$\vec{I} = 6 \angle -40° \text{ A}$$

Example: $v(t) = -4 \sin(30t + 50°)$ V

$$= 4 \cos(30t + 140°) \text{ V}$$

$$\vec{V} = 4 \angle 140° \text{ V}$$

$\cos\theta = \sin(\theta + 90)$
$\sin\theta = -\cos(\theta + 90)$
$\sin\theta = \cos(\theta - 90)$
$\cos\theta = -\sin(\theta - 90)$

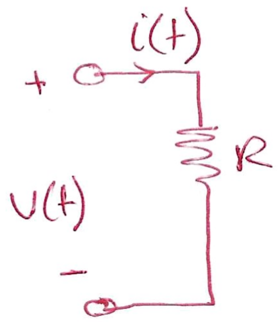
Example: If $y_1 = 20 \cos(\omega t - 30°)$ and $y_2 = 40 \cos(\omega t + 60°)$, express $y = y_1 + y_2$ as a single Sinusoidal function.

$\vec{Y_1} = 20 \angle -30°$
$\vec{Y_2} = 40 \angle 60°$

$$\vec{Y} = \vec{Y_1} + \vec{Y_2} = 20\angle -30° + 40 \angle 60°$$
$$= (17.32 - j10) + (20 + j34.64)$$
$$= 37.32 + j24.64 = 44.72 \angle 33.43°$$

$$\therefore y = 44.72 \cos(\omega t + 33.43°)$$

9.4 The pasive Circuit Elements in the Frequency Domain



*Resistor

$$v(t) = R\,i(t)$$

$$V_m e^{j(\omega t+\theta_v)} = R\, I_m e^{j(\omega t+\theta_i)}$$

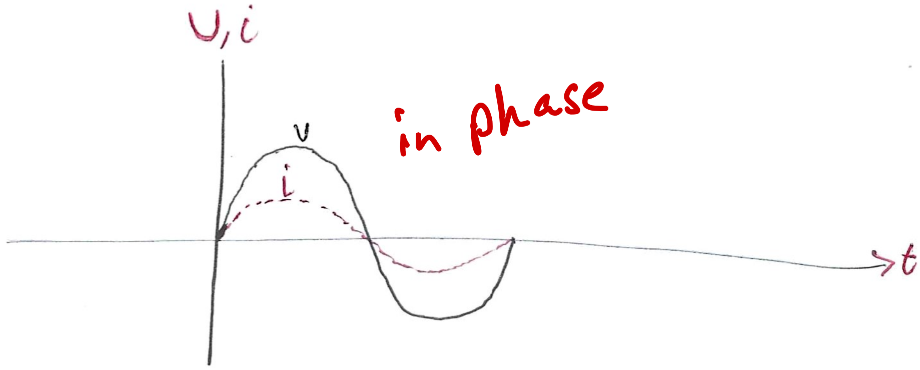
$$V_m e^{j\theta_v} = R\, I_m e^{j\theta_i}$$

$$V_m \angle\theta_v = R\, I_m \angle\theta_i$$

$$V_m = R\, I_m$$

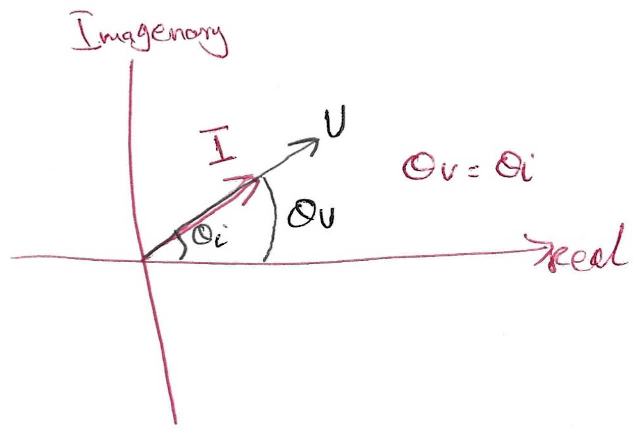
$$\angle\theta_v = \angle\theta_i$$

$$\vec{V} = R\,\vec{I}$$

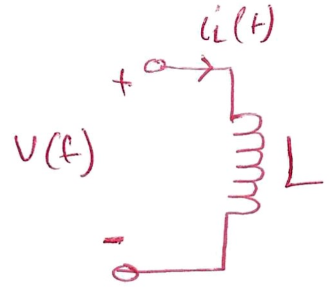


in phase

* Voltage and Current of a resistor are in phase



* Inductor :



$$V(t) = L\,\frac{di(t)}{dt}$$

$$V_m\, e^{j(\omega t+\theta_v)} = L\,\frac{d}{dt}\left(I_m\, e^{j(\omega t+\theta_i)}\right)$$

$$V_m\, e^{j(\omega t+\theta_v)} = j\omega L\, I_m\, e^{j(\omega t+\theta_i)}$$

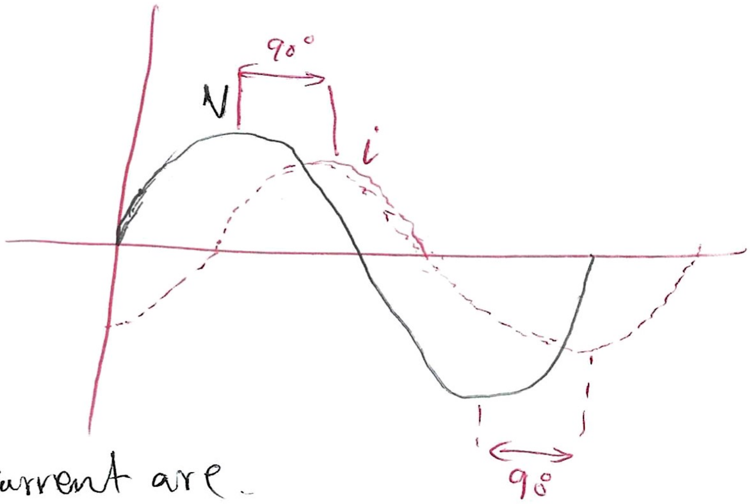
$$V_m\, e^{j\theta_v} = j\omega L\, I_m\, e^{j\theta_i}$$

$$V_m \angle\theta_v = j\omega L\, I_m \angle\theta_i = \omega L\, I_m \angle\theta_i+90$$

$$\vec{V} = j\omega L\, \vec{I}$$

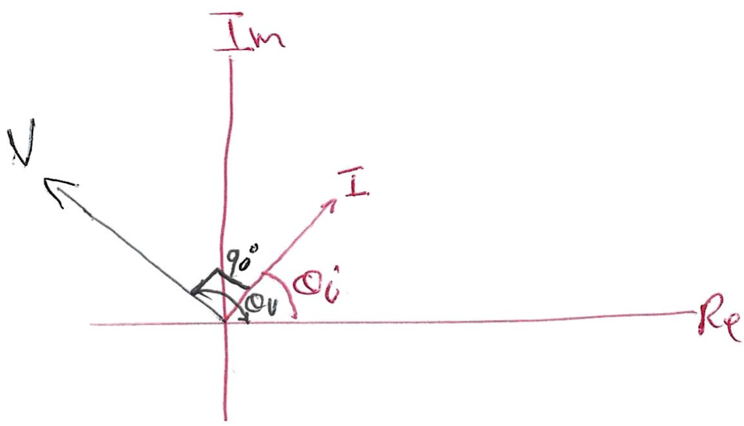
$$V_m = \omega L\, I_m$$

$$\theta_v = \theta_i + 90$$

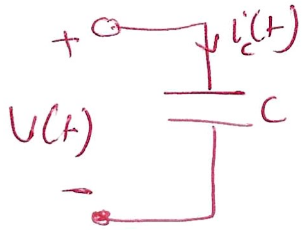


The sinusoidal voltage and current are out of phase of 90°

The Voltage leads the current by 90° and the current lags the voltage by 90°



* Capacitor :



$$i(t) = C\,\frac{dv(t)}{dt}$$

$$I_m e^{j(\omega t+\theta_i)} = C\,\frac{d}{dt}\left(V_m e^{j(\omega t+\theta_v)}\right)$$

$$I_m\, e^{j(\omega t+\theta_i)} = j\omega C\, V_m\, e^{j(\omega t+\theta_v)}$$

$$I_m\, e^{j\theta_i} = j\omega C\, V_m\, e^{j\theta_v}$$

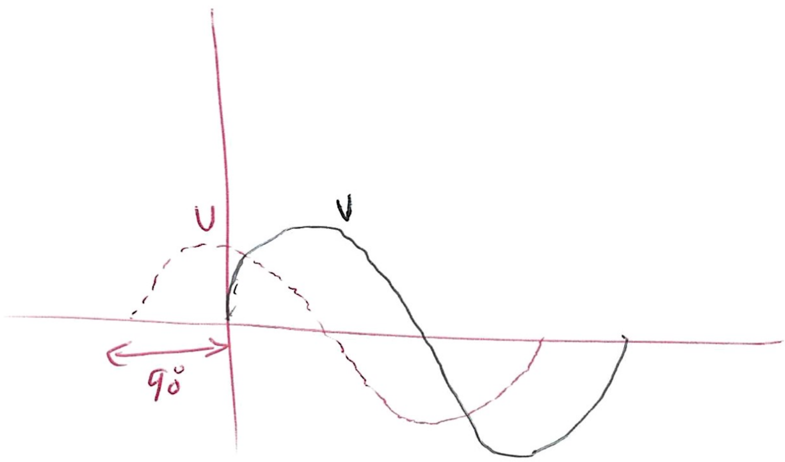
$$I_m \angle\theta_i = j\omega C\, V_m \angle\theta_v$$

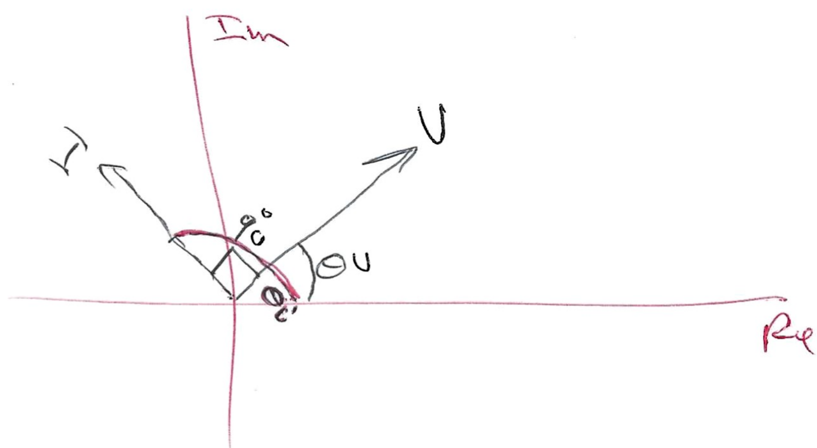
$$= \omega C\, V_m \angle\theta_v+90$$

$$\vec{I} = j\omega C\, \vec{V}$$

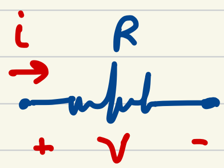
$$I_m = \omega C\, V_m \quad \Rightarrow \quad V_m = I_m\left(\frac{1}{\omega C}\right)$$

$$\theta_i = \theta_v + 90^\circ \qquad \theta_v = \theta_i - 90^\circ$$

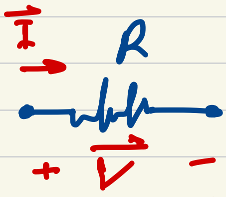




Passive Elements in phasor-domain :

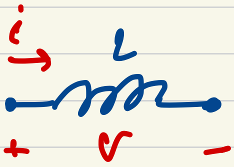
1) 

$\xrightarrow[\text{transform}]{P}$ phasor transform

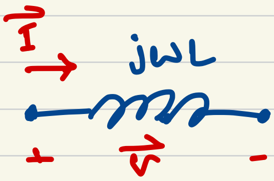


$v = Ri$ $\qquad\qquad$ $\vec{V} = R\vec{I}$

i + v are in-phase

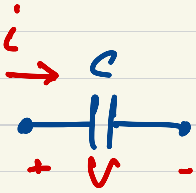
2) 

$\xrightarrow{P}$

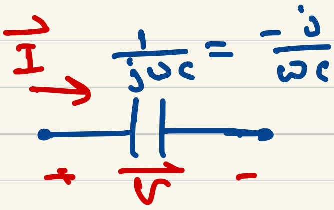


i lags v by $90°$

$v = L\dfrac{di}{dt}$ $\qquad$ $\vec{v} = (j\omega L)\,\vec{I}$ $\qquad$ $\theta_v = \theta_i + 90$

3) 

$\xrightarrow{P}$



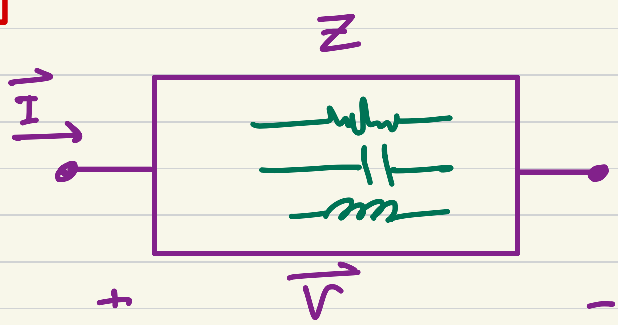
i leads v by $90°$

$\theta_i = \theta_v + 90$

$i = c\dfrac{dv}{dt}$

$$\vec{v} = \left(\frac{1}{j\omega c}\right)\vec{I}$$
$$= \left(\frac{-j}{\omega c}\right)\vec{I}$$

$\boxed{\text{Impedance}}$



$$\vec{V} = \underset{\substack{\downarrow \\ \text{impedance}}}{Z}\,\vec{I}$$

The impedance is measured in [Ω]

Impedance, Reactance and Admitance

The relation between the voltage and current on the phasor domain (Complex or frequency) for the three elements

$$\vec{V}_R = R\,\vec{I} \qquad \vec{V}_L = j\omega L\,\vec{I} \qquad \vec{V}_C = \frac{1}{j\omega C}\,\vec{I}$$

$$V = Z\,I$$

$Z_R = R$ — Impedance of a resistor: $a = R$, $b = 0$

$Z_L = j\omega L$ — impedance of an Inductor: $a = 0$, $b = \omega L$

$Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}$ — Impedance of a Capacitor: $a = 0$, $b = \frac{-1}{\omega C}$

$Z = a + jb$

The imaginary part of the impedance is called Reactance

$$\boxed{\begin{array}{l} Z = a + jb \\ a : \text{Resistive} \\ b : \text{Reactance} \end{array}}$$

Reactance of a risistor: $X_R = 0$

Reactance of an inductor: $X_L = \omega L$

Reactance of a Capacitor: $X_C = \frac{-1}{\omega C}$

The Admitance is the reciprocal of the impedance

$$Y = \frac{1}{Z}$$

$$Y_R = \frac{1}{Z_R} = \frac{1}{R}$$

$$Y_L = \frac{1}{Z_L} = \frac{1}{j\omega L} = -j\frac{1}{\omega L}$$

$$Y_C = \frac{1}{Z_C} = \frac{1}{\frac{1}{j\omega C}} = j\omega C$$

$$Z(j\omega) = \frac{\vec{V}}{\vec{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i}$$

$$= \frac{V_m}{I_m} \angle \theta_v - \theta_i$$

$$= |Z| \angle \theta_z$$

$$= R + jX$$

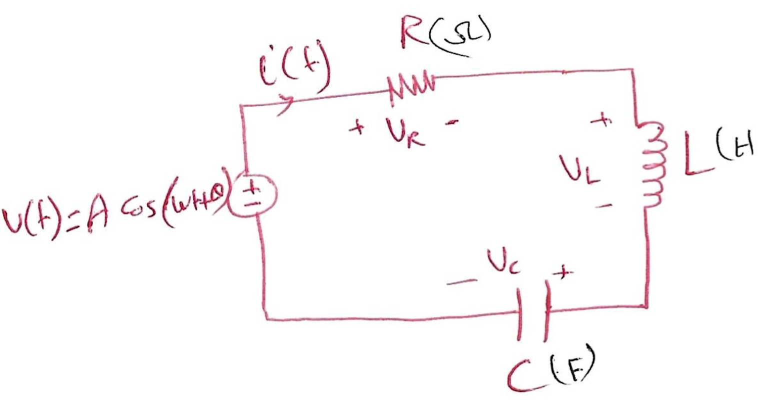
$R \equiv$ Resistive part

$X \equiv$ Reactive part

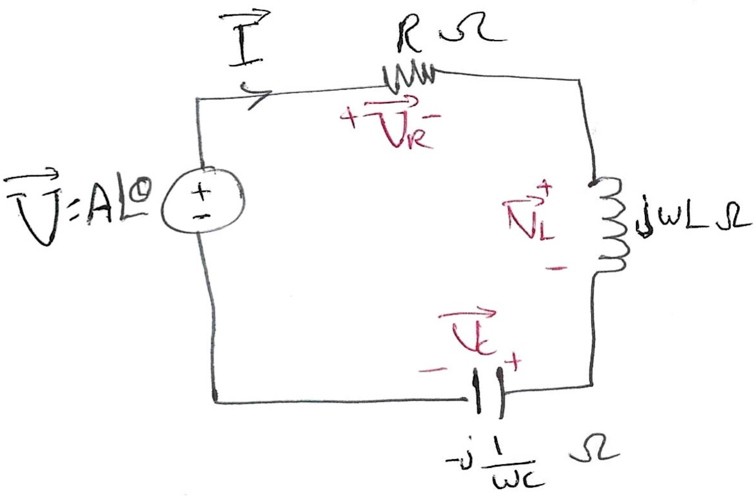
$$|Z| = \sqrt{R^2 + X^2} \qquad R = |Z| \cos(\theta_z)$$

$$\theta_z = \tan^{-1}\left(\frac{X}{R}\right) \qquad X = |Z| \sin(\theta_z)$$

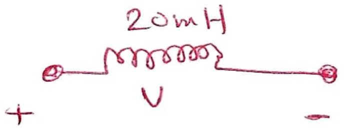
Time domain



phasor (Complex, frequency) domain



Example



$i(t) = 10\cos(10{,}000t + 30°)$ mA

Calculate:-

a) The inductive Reactance:-

$$\omega L = (10^4)(20 \times 10^{-3}) = 200\,\Omega$$

b) The impedance of the inductor:-

$$Z_L = j\omega L = j200\,\Omega$$

$$Z_1 * Z_2 = (|Z_1|\angle\theta_1)\,|Z_2|\angle\theta_2 = Z_1 Z_1 \angle\theta_1+\theta_2$$

$$\frac{Z_1}{Z_2} = \frac{|Z_1|\angle\theta_1}{|Z_2|\angle\theta_2} = \frac{Z_1}{Z_2}\angle\theta_1-\theta_2$$

c) The phasor Voltage $\vec{V}$

$$\vec{V}_L = \vec{I} Z_L = (10\angle 30)(200\angle 90) \times 10^{-3} = 2\angle 120°\ \text{V}$$

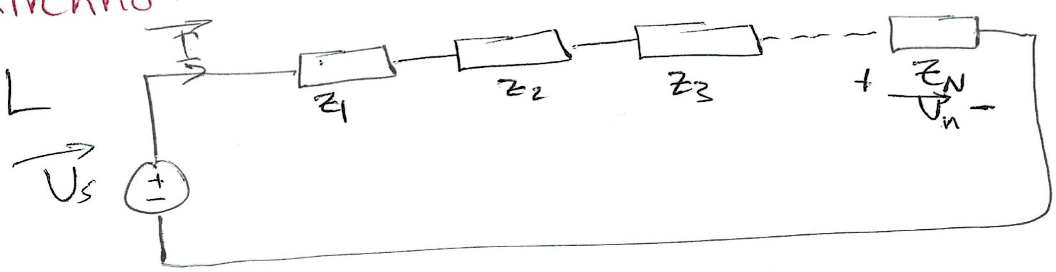
mA

d) the steady state expression for $V(t)$

$$V_L(t) = 2\cos(10{,}000t + 120°)\ \text{V}$$

9.5 Kirchhoff's Laws in the Frequency Domain:-

KVL



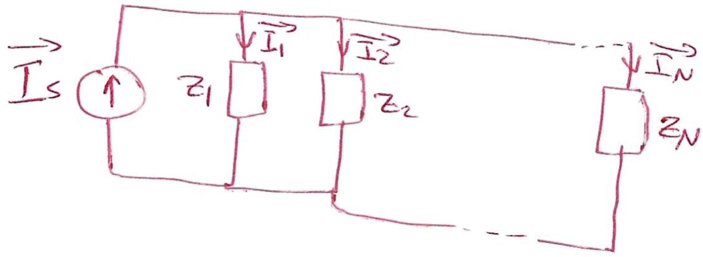
$$\vec{V}_s = \vec{V}_1 + \vec{V}_2 + \vec{V}_3 + \dots + \vec{V}_n$$

$$Z_{eq} = Z_1 + Z_2 + Z_3 + \dots + Z_N$$

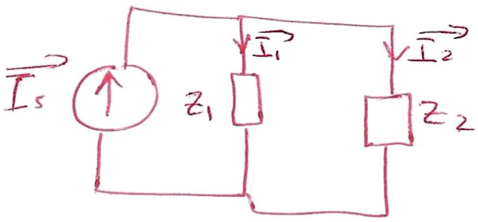
$$\vec{V}_n = \frac{Z_n}{Z_1 + Z_2 + \dots + Z_N} \cdot \vec{V}_s$$

KCL

$$\vec{I_s} = \vec{I_1} + \vec{I_2} + \cdots + \vec{I_N}$$



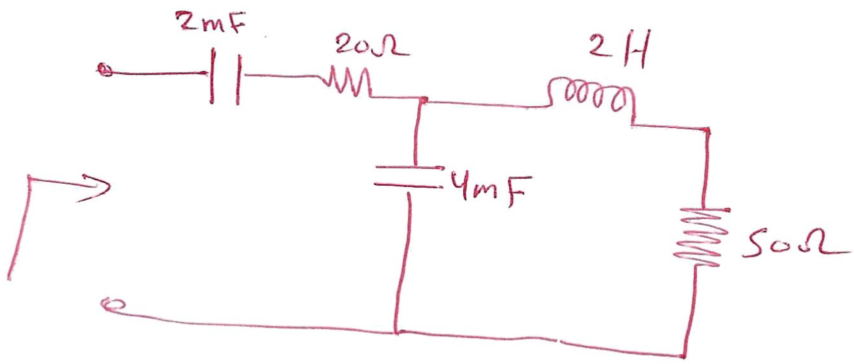
$$\vec{I_1} = \frac{Z_2}{Z_1 + Z_2}\vec{I_s}$$



Example: find Z_{eq}

given that $\omega = 10$ rad/s.

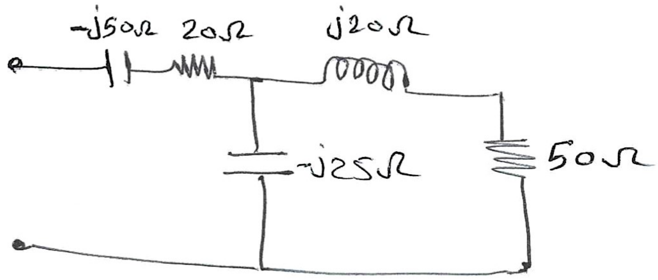
Z_{eq}



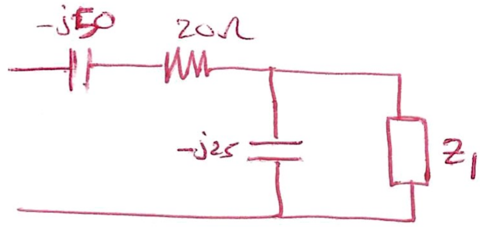
$$Z_{2mF} = -j\left(\frac{1}{10 \times 2 \times 10^{-3}}\right) = -j50\,\Omega$$

$$Z_{2H} = j(10)(2) = j20\,\Omega$$

$$Z_{4mF} = -j\left(\frac{1}{10 \times 4 \times 10^{-3}}\right) = -j25\,\Omega$$



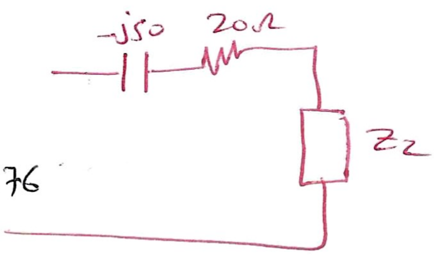
$$Z_1 = 50 + j20$$



$$Z_2 = (-j25 \,//\, Z_1)$$

$$= \frac{-j25(50 + j20)}{50 + j20 - j25}$$

$$= \frac{-j1250 + 500}{50 - j5} = 12.38 - j23.76$$

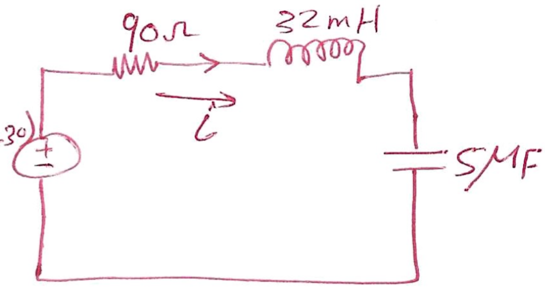


$$Z_{eq} = -j50 + 20 + Z_2$$

$$= 32.38 - j73.76\,\Omega$$

Example:- For the circuit shown below, the source voltage is sinusoidal

$V_s(t) = 750\cos(5000t+30)$

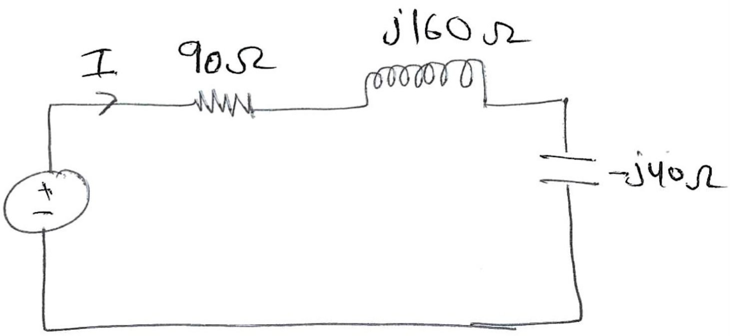


a) construct the frequency-domain equivalent circuit?

b) calculate the steady state current $i(t)$

$Z_L = j\omega L = j(5000)(32\times10^{-3}) = j160\,\Omega$

$Z_C = -j\left(\frac{1}{\omega C}\right) = -j\left(\frac{1}{(5000)(5\times10^{-6})}\right) = -j40\,\Omega$



$\vec{I} = \frac{\vec{V}}{Z} = \frac{750\angle 30}{90+j160-j40} = \frac{750\angle 30}{90+j120}$

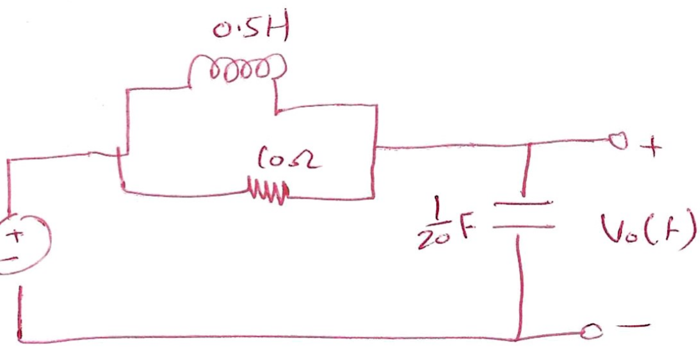
$= \frac{750\angle 30}{150\angle 53.13} = 5\angle -23.13\ A$

$\theta_v - \theta_i$

$\therefore i(t) = 5\cos(5000t - 23.13^\circ)\ A$

Example:- Calculate $V_o(t)$

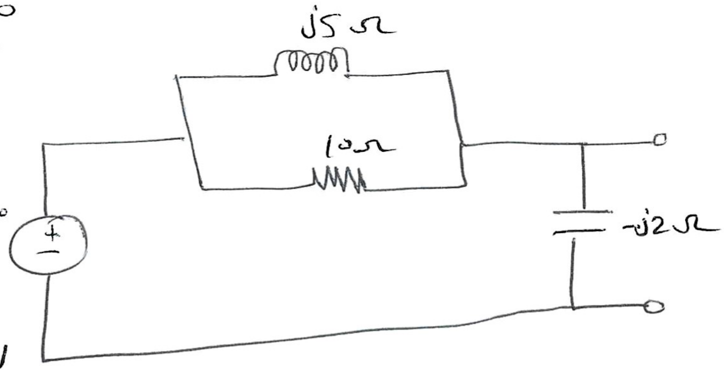
$V_s(t) = 10\cos(10t+75)$



$\vec{V_o} = \frac{-j2}{-j2 + (10 \parallel j5)} \cdot 10\angle 75^\circ$

$\vec{V_o} = 7.071\angle -60^\circ\ V$

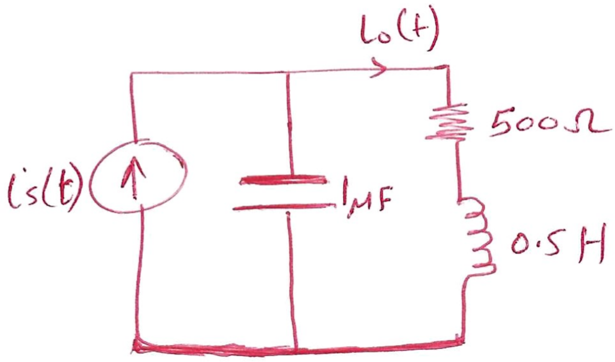
$10\angle 75^\circ$



$V_o(t) = 7.07\cos(10t - 60^\circ)\ V$

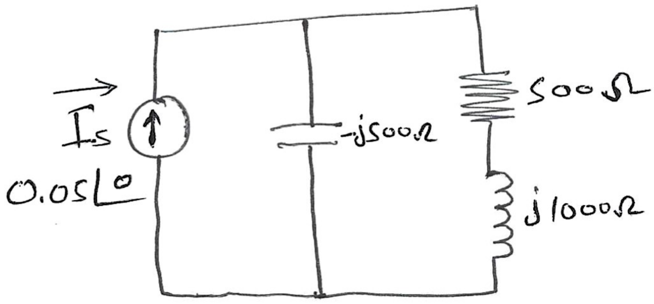
Example :- Calculate $i_o(t)$

$i_s(t) = 0.05 \cos(2000t)$ A



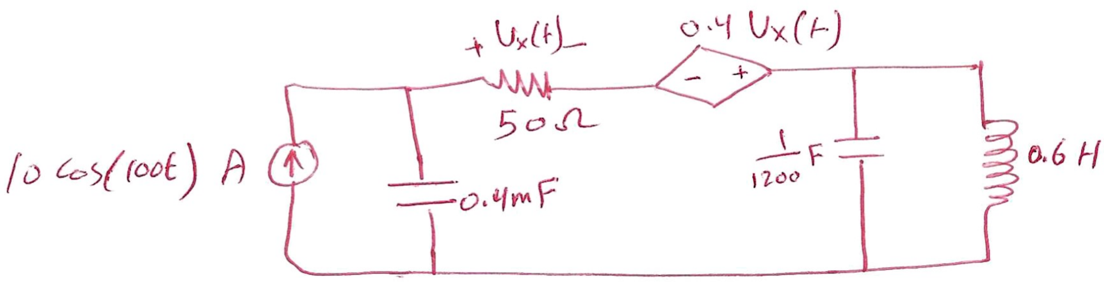
$$I_o = \frac{-j500}{-j500 + j1000 + 500}(0.05\angle 0)$$

$$= 0.03535 \angle -45^\circ \quad -135$$



$\therefore i_o(t) = 0.03535 \cos(2000t - 45^\circ)$ A

Example : For the Circuit shown, find $V_x(t)$?

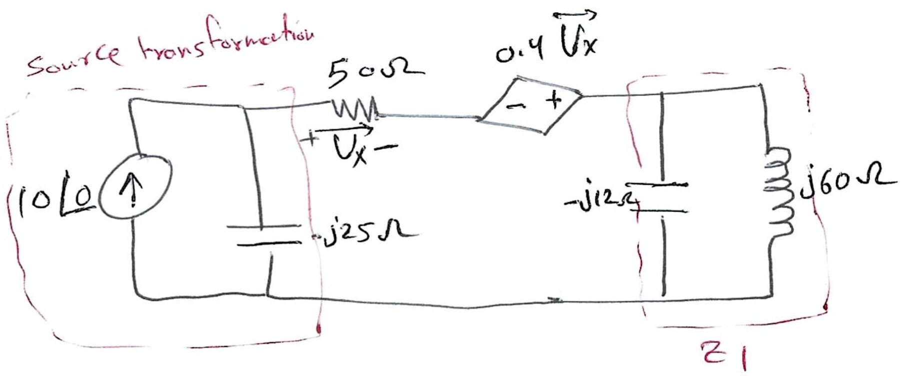


$Z_1 = -j12 \,//\, j60$

$$= \frac{(-j12)(+j60)}{j60 - j12}$$

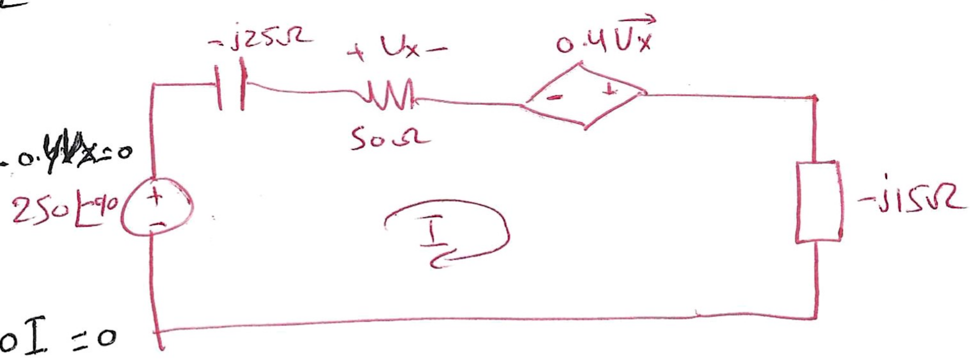
$$= \frac{720}{j48} = \frac{720\angle 0}{48\angle 90}$$

$$= -j15\ \Omega$$



KUL

$-250\angle 90 + I(-j15 - j25 + 50) - \cancel{0.4V_x} = 0$



$V_x = 50I$

$-250\angle 0 + 50I - 20I - j40I = 0$

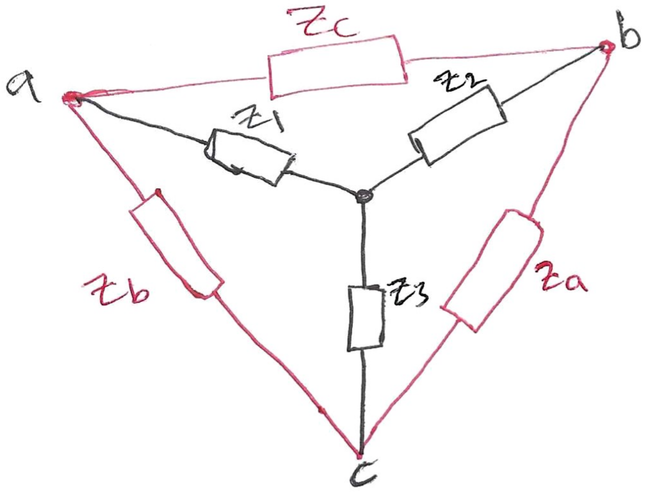
$I = \frac{250\angle 0^\circ}{30 - j40} = 5\angle 36.87$ A

$$\vec{V_x} = 50\,\vec{I}$$

$$= (50\angle 0)(5\angle -36.87^\circ) = 250\angle -36.87\ \text{V}$$

$$V_x(t) = 250\cos(100t - 36.87^\circ)\ \text{V}$$

Delta to Wye Transformation



Δ to Y

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

Y to Δ

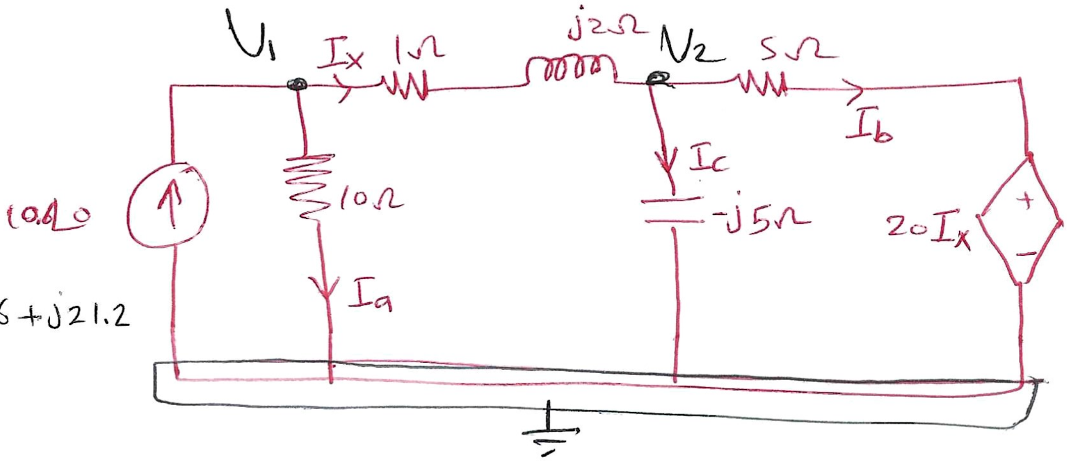
$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$

Nodal Analysis

Example 8- Use the node Voltage method to find the branch Currents I_a, I_b and I_c in the circuit shown



at node ①

$$-10.6+\frac{V_1}{10}+\frac{V_1-V_2}{1+j2}=0$$

$$V_1(1.1+j0.2)-V_2=10.6+j21.2$$

at node ②

$$\frac{V_2-V_1}{1+j2}+\frac{V_2}{-j5}+\frac{V_2-20I_x}{5}=0 \qquad \text{Since} \qquad I_x=\frac{V_1-V_2}{1+j2}$$

$$-5V_1+(4.8+j0.6)V_2=0$$

$$\Rightarrow V_1=68.4-j16.8 \quad V$$

$$V_2=68-j26 \quad V$$

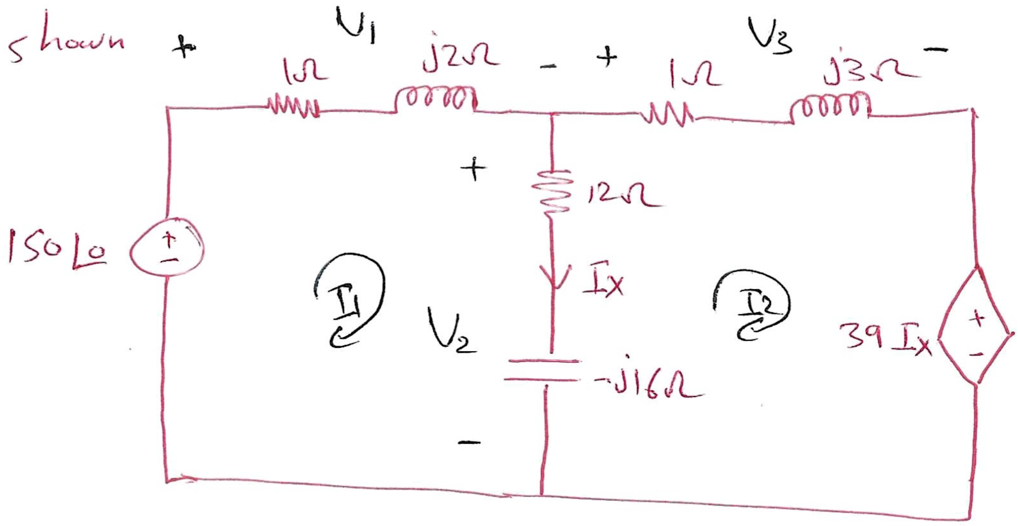
$$I_a=\frac{V_1}{10}=6.84-j1.68 \text{ A}$$

$$I_b=\frac{V_2-20I_x}{5}=-1.44-j11.92 \text{ A}$$

$$I_c=\frac{V_2}{-j5}=5.2+j13.6 \text{ A}$$

$$I_x=\frac{V_1-V_2}{1+j2}=3.76+j1.68 \text{ A}$$

Example :- use the mesh current method to find the voltages V_1, V_2 and V_3 in the circuit shown



for mesh ①

$$-150\angle 0 + I_2(1+12+j2-j16) - I_2(12-j16) = 0$$

for mesh ②

$$I_2(1+12+j3-j16) - I_1(12-j16) + 39\,I_x = 0$$

$$I_x = I_1 - I_2$$

solving for I_1 and I_2

$$I_1 = -26 - j52\ A$$

$$I_2 = -24 - j58\ A$$

$$I_x = I_1 - I_2 = -2 + j6\ A$$

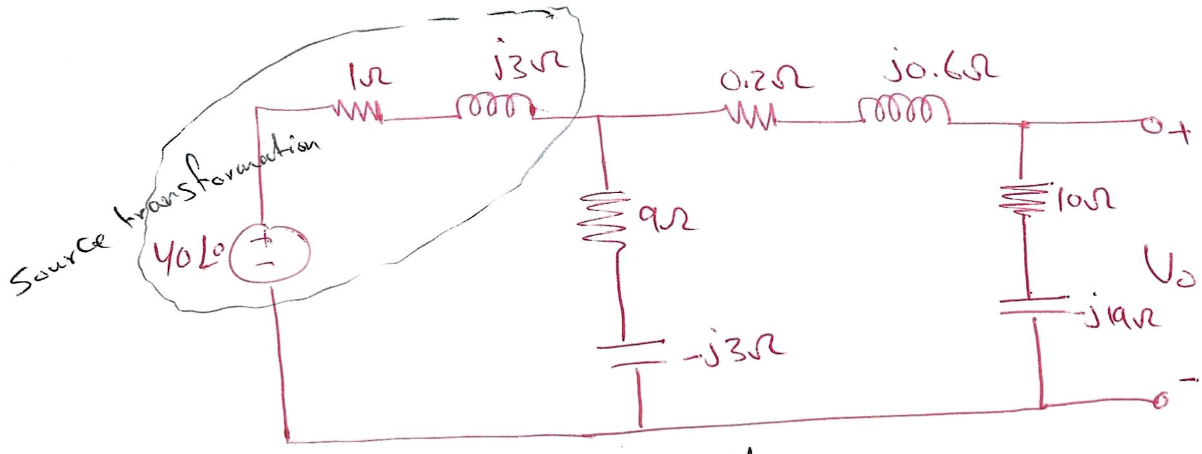
$$V_1 = (1+j2)\,I_1 = 78 - j104\ V$$

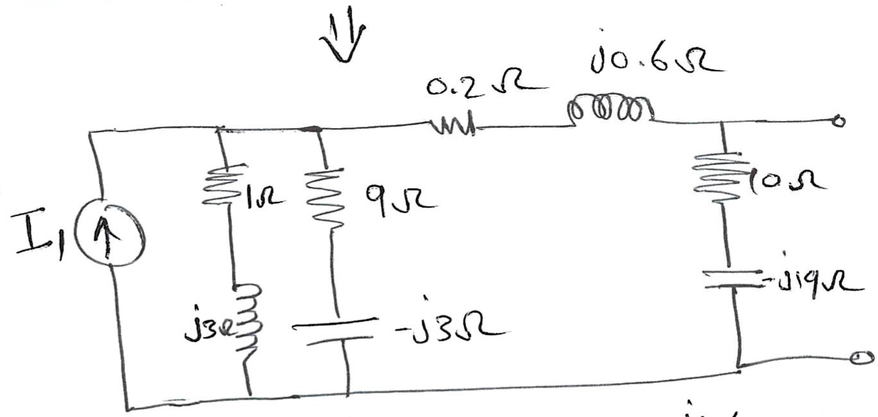
$$V_2 = (12-j16)\,I_x = 72 + j104\ V$$

$$V_3 = (1+j3)\,I_2 = 150 - j130\ V$$

* Source Transformation

Example:- use the concept of source transformation to find the phasor Voltage V_o in the circuit shown

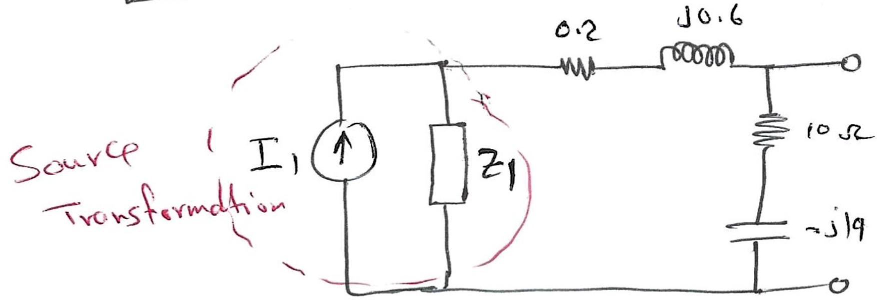




$$I_1 = \frac{40\angle 0}{1+j3\Omega} = \frac{40\angle 0}{3.16\angle 71.565}$$

$$= 12.658\angle -71.565$$

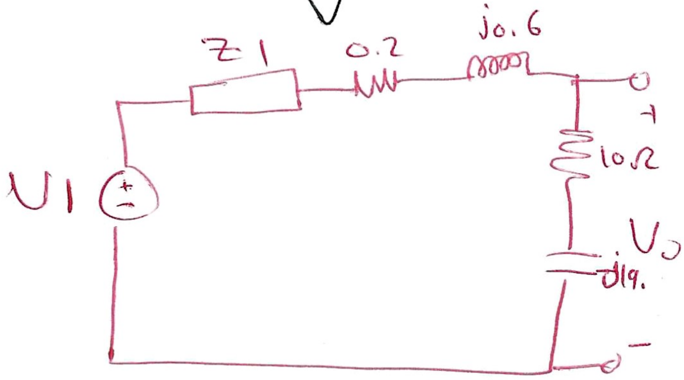
$$= 4 - j12$$



$$Z_1 = \frac{(1+j3)(9-j3)}{1+9+j3-j3}$$

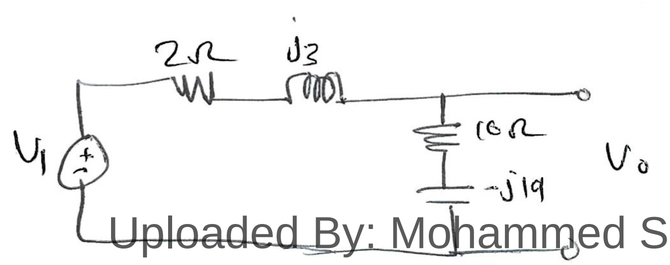
$$= \frac{9-j3+j27+9}{10}$$

$$= \frac{18+j24}{10} = 1.8+j2.4$$



$$V_1 = I_1 Z_1 = (4-j12)(1.8+j2.4)$$

$$= 36 - j12\ V$$



$$V_o = \frac{(10-j19)\ V_1}{2+j3+10-j19}$$

$$= 36.12 - j18.84\ V$$

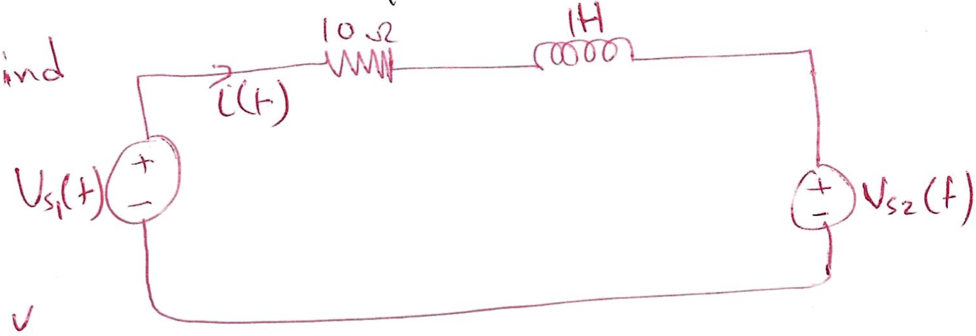
The power of superposition

Superposition is the only method to analyse a circuit of two sources with 2 different frequencies

For the circuit shown, find $i(t)$

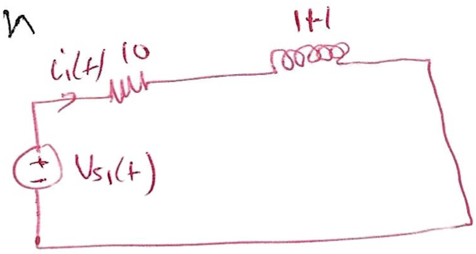
$V_{s1}(t) = 100\cos(10t)$

$V_{s2}(t) = 50\cos(20t - 10^\circ)$ V



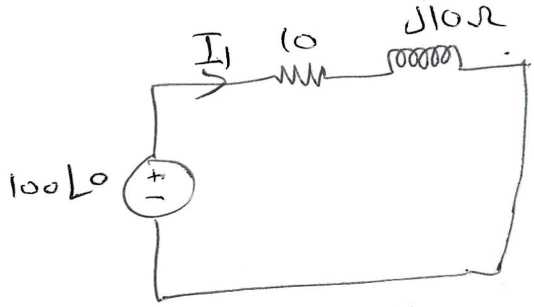
$$i(t) = i_1(t) + i_2(t)$$

① Let $V_{s2}(t)$ off, and $V_{s1}(t)$ on

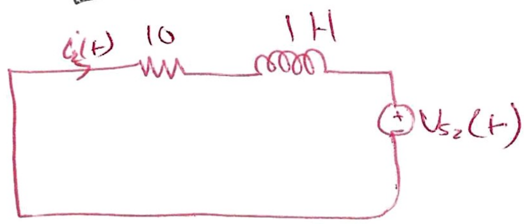


$$I_1 = \frac{100\angle 0}{10 + j10} = 7.07\angle -45^\circ \text{ A}$$

$$i_1(t) = 7.07\cos(10t - 45^\circ) \text{ A}$$



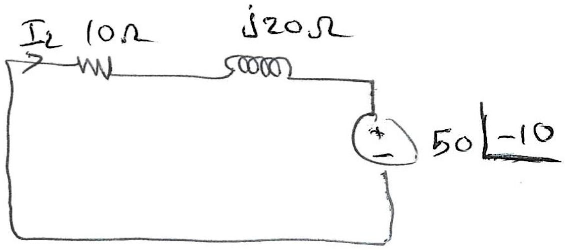
② Let $V_{s1}(t)$ off and $V_{s2}(t)$ on



$$I_2 = \frac{-50\angle -10^\circ}{10 + j20} = \frac{50\angle 170}{22.36\angle 63.43}$$

$$= 2.24\angle 106.57^\circ \text{ A}$$

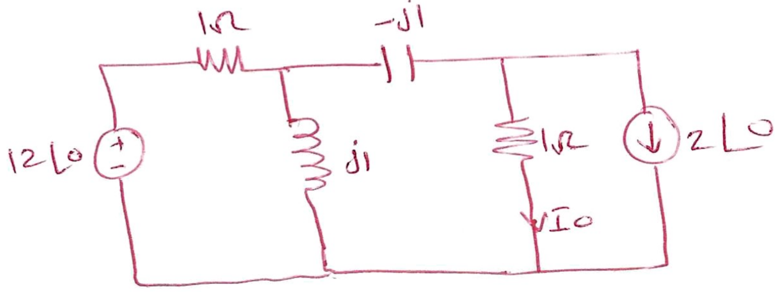
$$i_2(t) = 2.24\cos(20t + 106.57^\circ)$$



$$\therefore i(t) = 7.07\cos(10t - 45^\circ) + 2.24\cos(20t + 106.57^\circ) \text{ A}$$

Thevenin's and Norton's theorems

Example: Find I_o using Thevenin's theorem



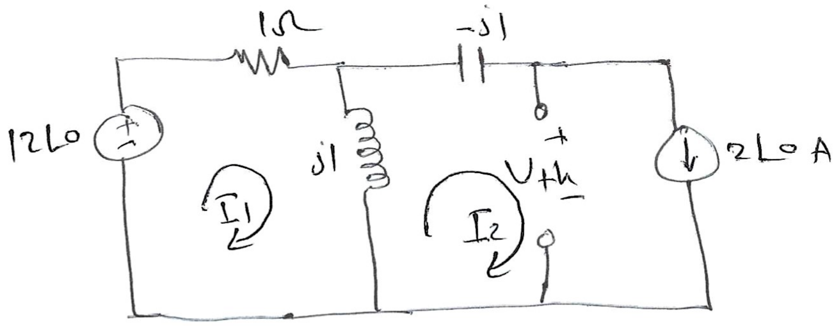
$$I_o = \frac{V_{th}}{Z_{th} + 1\Omega}$$

1) To find V_{th}

$V_{th} + I_2(j1 - j1) - I_1(j1) = 0$

$\Rightarrow V_{th} = j I_1$

$I_2 = 2\angle 0 \text{ A}$



For mesh (1)

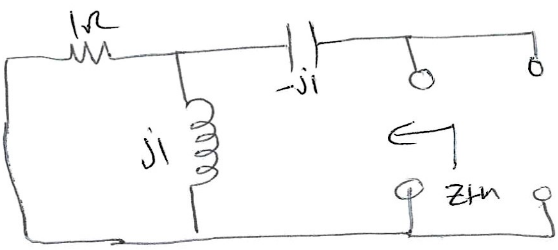
$-12\angle 0 + I_1(1+j) - I_2(j) = 0$

$-12\angle 0 + I_1(1+j) - 2\angle 0(j) = 0$

$$\therefore I_1 = \frac{12 + j2}{1 + j} \text{ A}$$

$$\Rightarrow V_{th} = \left(\frac{-2 + j12}{1 + j}\right) V = 5 + j7$$

2) to find Z_{th}, set all the independent sources to zero



$Z_{th} = -j1 + (1 \parallel j1)$

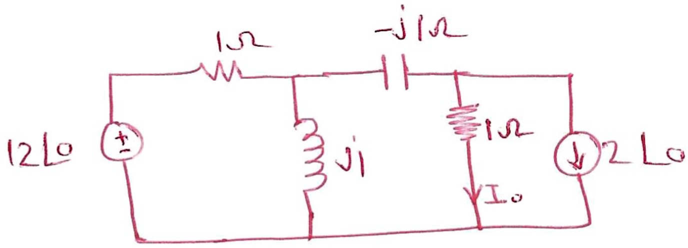
$= -j1 + \frac{1}{2} + j\frac{1}{2}$

$= \frac{1}{2} - j\frac{1}{2}$

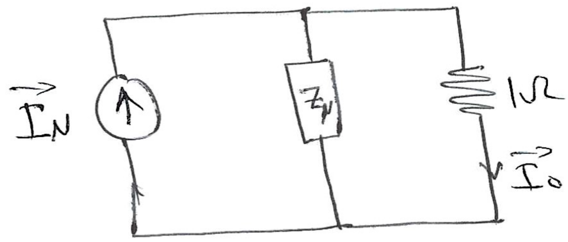
$$\therefore I_o = \frac{V_{th}}{Z_{th} + 1\Omega} = \frac{5 + j7}{1.5 - j\frac{1}{2}} = \left(\frac{8}{5} + j\frac{26}{5}\right) A$$

Norton's theorem

Find $\vec{I_o}$ using Norton's theorem

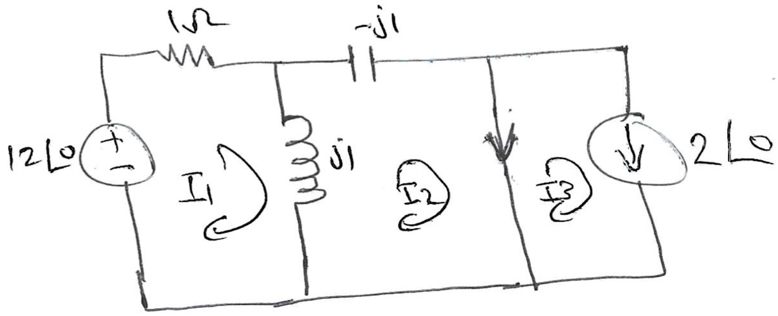


$$I_o = I_N \frac{Z_N}{Z_N + 1\Omega}$$



$I_N = I_2 - I_3$

$I_3 = 2\angle 0 \text{ A}$



for mesh ①

$-12 + I_1(1+j) - I_2(j) = 0$

for mesh ②

$I_2(j-j) - I_1(j) = 0$

$-jI_1 = 0$

$\therefore I_1 = 0$

$\Rightarrow I_2 = 12\angle 90 \text{ A}$

$\therefore I_N = I_2 - I_3 = -2 + j12$

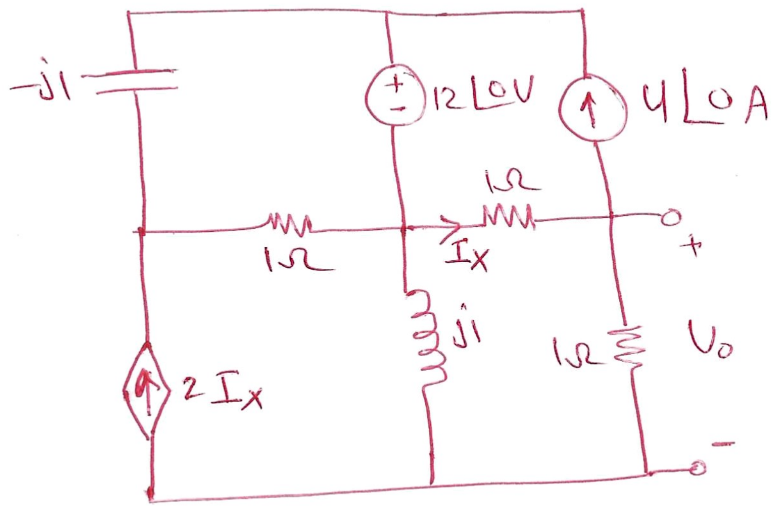
$Z_N = Z_{th} = \frac{1}{2} - j\frac{1}{2}$

$$I_o = \frac{I_N Z_N}{Z_N + 1}$$

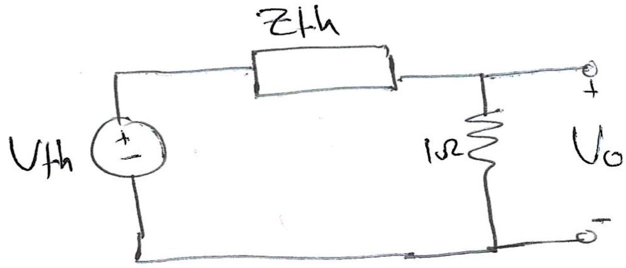
$$= \frac{8}{5} + j\frac{26}{5} \text{ A}$$

Example:- for the circuit shown, find V_o using Thevenin's theorem

$$V_o = \frac{1}{1+Z_{th}} V_{th}$$



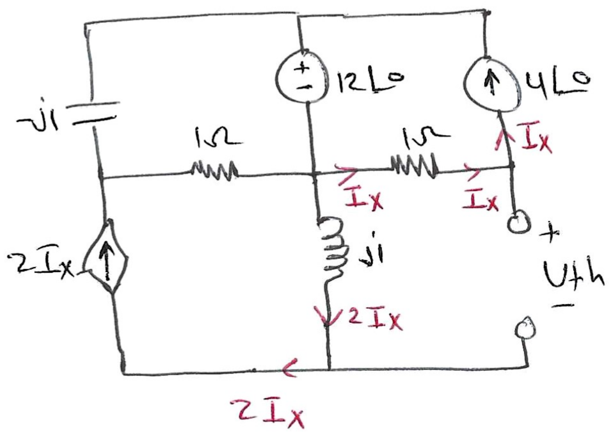
① To find V_{th}



$$V_{th} = (-1\Omega)(I_x) + j1\Omega(2I_x)$$

$$I_x = 4\angle 0 \ A$$

$$\therefore V_{th} = -4 + j8 \ A$$

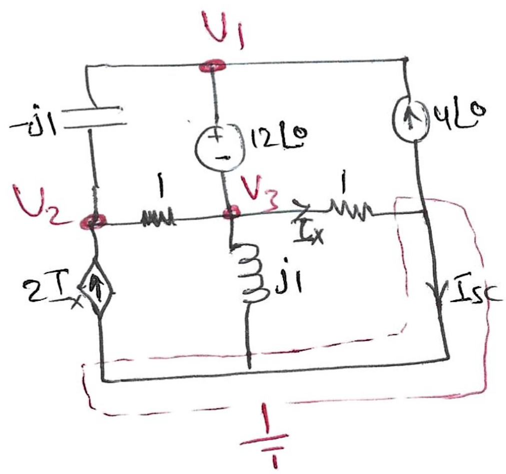


② to find Z_{th}

$$Z_{th} = \frac{V_{th}}{I_{sc}}$$

$$I_{sc} = I_x - 4\angle 0$$

$$\vec{I_x} = \frac{\vec{V_3}}{1\Omega} = \vec{V_3}$$



node 1 and node 3 are supernodes

$$V_1 - V_3 = 12\angle 0$$

$$-4\angle 0 + \frac{V_1 - V_2}{-j} + \frac{V_3 - V_2}{1} + \frac{V_3}{j} + \frac{V_3}{1} = 0$$

at node (2)

$$-2I_x + \frac{V_2 - V_1}{-j} + \frac{V_2 - V_3}{1} = 0$$

Solving for V_1, V_2 and V_3

$$V_3 = \frac{4j}{1-j}$$

$$\therefore I_{sc} = -\left(\frac{8+j4}{1+j1}\right)$$

$$\therefore Z_{th} = \frac{V_{th}}{I_{sc}} = \underline{1-j}$$ Capacitive

$$\Rightarrow V_o = \frac{-4+j8}{1+1-j}(1) = 4\angle 143.13^\circ \, V$$