

10.10 The Binomial Series and Applications of Taylor Series

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* The Binomial series of $f(x) = (1+x)^m$ is "using Taylor series expa."

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k, \quad |x| < 1, \quad m \text{ is constant}$$

where the series converges absolutely.

where $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2}$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!} \quad \text{for } k \geq 3$$

Powers and Roots

Exp Find the first four terms of the binomial series for

$$\begin{aligned} \text{[1]} \quad (1+x)^{\frac{1}{2}} &= 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} x^k = 1 + \frac{x}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})x^2}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})x^3}{3!} + \dots \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots \end{aligned}$$

$$\begin{aligned} \text{[2]} \quad \left(1 - \frac{x}{2}\right)^{-2} &= 1 + \sum_{k=1}^{\infty} \binom{-2}{k} \left(\frac{-x}{2}\right)^k \\ &= 1 + x + \frac{(-2)(-3)\left(\frac{-x}{2}\right)^2}{2} + \frac{(-2)(-3)(-4)\left(\frac{-x}{2}\right)^3}{3!} + \dots \\ &= 1 + x + \frac{3x^2}{4} + \frac{x^3}{2} + \dots \end{aligned}$$

Approximating Nonelementary Integrals

Exp Use series to estimate the following integrals with an error of magnitude less than 10^{-3}

$$\begin{aligned} \text{[1]} \quad \int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} &= \int_0^{0.1} (1+x^4)^{-\frac{1}{2}} dx = \int_0^{0.1} \left(1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} x^{4k}\right) dx \\ &= \int_0^{0.1} \left(1 - \frac{x^4}{2} + \frac{3x^8}{8} - \dots\right) dx = \left[x - \frac{x^5}{10} + \frac{3x^9}{9(8)} - \dots\right]_0^{0.1} \\ &\approx x \Big|_0^{0.1} \approx 0.1 \quad \text{with error } |E| \leq \frac{(0.1)^5}{10} \approx 0.000001. \end{aligned}$$

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$$\begin{aligned} \boxed{2} \int_0^{0.2} \frac{e^{-x} - 1}{x} dx &= \int_0^{0.2} \frac{1}{x} (e^{-x} - 1) dx \\ &= \int_0^{0.2} \frac{1}{x} \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots - 1 \right) dx \\ &= \int_0^{0.2} \left(-1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \dots \right) dx \\ &= \left[-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{4(4!)} - \dots \right]_0^{0.2} \approx -(0.2) + \frac{(0.2)^2}{4} - \frac{(0.2)^3}{18} \approx -0.19044 \\ &\text{with error } |E| \leq \frac{(0.2)^4}{96} \approx 0.00002 \end{aligned}$$

Arctangents: Remember that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

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with Leibniz's formula:

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots + \frac{(-1)^n}{2n+1} + \dots$$

• Note that $\tan^{-1} x = \int \frac{dx}{1+x^2}$

$$\Rightarrow \frac{1}{1+x^2} = \frac{d}{dx} \tan^{-1} x = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}$$

• Integrate both sides from 0 to x \Rightarrow

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \underbrace{\int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt}_{R_n(x)}$$

where $|R_n(x)| \leq \int_0^{|x|} t^{2n+2} dt = \frac{|x|^{2n+3}}{2n+3} \rightarrow 0$ as $n \rightarrow \infty$ if $|x| \leq 1$

• If $|x| \leq 1$, then $\lim_{n \rightarrow \infty} R_n(x) = 0$ and so

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \quad |x| \leq 1.$$

Q18 Find Taylor series of $f(x) = \ln x$ at $x=1$ 82.1

$$f(x) = \ln x \quad \Rightarrow f(1) = 0$$

$$f'(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow f'(1) = 1$$

$$f''(x) = -x^{-2} \quad \Rightarrow f''(1) = -1$$

$$f'''(x) = 2x^{-3} \quad \Rightarrow f'''(1) = 2$$

$$f^{(4)}(x) = -6x^{-4} \quad \Rightarrow f^{(4)}(1) = -6$$

⋮

$$\ln x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n$$

$$= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots$$

$$= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4 + \dots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-1)^n}{n}$$

Indeterminate forms

Exp Use series to evaluate the limits:

$$\boxed{1} \lim_{x \rightarrow 1} \frac{\ln x}{x-1}$$

Taylor series of $\ln x$ about $x=1$ is

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \dots$$

$$= \lim_{x \rightarrow 1} \left(1 - \frac{1}{2}(x-1) + \dots \right) = 1$$

$$\boxed{2} \lim_{x \rightarrow 0} \frac{e^x - (1+x)}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x^2} (e^x - 1 - x)$$

$$= \lim_{x \rightarrow 0} \frac{1}{x^2} \left(1 + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{3!} + \dots - 1 - x \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{2} + \frac{x}{3!} + \dots \right) = \frac{1}{2}$$

$$\boxed{3} \lim_{y \rightarrow 0} \frac{y - \tan^{-1} y}{y^3} = \lim_{y \rightarrow 0} \frac{1}{y^3} \left[y - \left(y - \frac{y^3}{3} + \frac{y^5}{5} - \dots \right) \right]$$

$$= \lim_{y \rightarrow 0} \left[\frac{1}{3} - \frac{y^2}{5} + \dots \right] = \frac{1}{3}$$

Euler's formula

$$i = \sqrt{-1}, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i$$

Recall that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right)$$

$$e^{i\theta} = \cos \theta + i \sin \theta \rightarrow \text{Euler's formula } \theta \text{ is polar angle.}$$

• Any complex number has the form $a + bi$, $a, b \in \mathbb{R}$.

Exp $e^{i\pi} = \cos \pi + i \sin \pi = -1$